

APPLIED MATH 9
Computation Handout 2

To generate a uniformly distributed random number on the interval (0,1) in matlab, we use the rand() function:

x=rand;

To generate an $n \times n$ matrix with random entries, we use

x=rand(n).

To take a minimum along each column, that is $\min_{i=1,\dots,m} a_{ij}$, set

xm=min(x).

This produces a row vector. To take a minimum along each row, that is $\min_{j=1,\dots,n} a_{ij}$

xm=min(x,[],2).

Similarly for the maximum

xma=max(x)

xma=max(x,[],2).

Let's show that the lower game is lower than the upper game with a numerical test: (input in matlab)

x=rand(3)

lower=min(x,[],2)

lower=max(lower)

or

lower=max(min(x,[],2))

For the upper game:

upper=max(x)

upper=min(upper)

or

upper=min(max(x))

Let's repeat it with

$$x=\text{rand}(7).$$

Determinants. For a 2 by 2 matrix, the determinant is defined as

$$\det(A) = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} \times a_{22} - a_{21} \times a_{12}.$$

For an $n \times n$ matrix A , it is defined inductively as

$$\det(A) = \sum_{j=1}^n a_{1j}(-1)^{1+j} \det(M_{1j}(A))$$

where $M_{ij}(A)$ is the $(n-1) \times (n-1)$ submatrix of A obtained by deleting row i and column j of A . The efficient way to compute a determinant is to use the row operations described below to reduce A to a what is called a “triangular” matrix T and then evaluate $\det(T)$. A matrix is triangular if either all entries above the diagonal are zero or if all entries below the diagonal are zero. The procedure uses the following properties of the determinant.

- $\det(T) = t_{11}t_{22} \cdots t_{nn}$ if T is a triangular matrix.
- if B is obtained from A by the row operation $(R_i \leftrightarrow R_j)$ (interchange rows i and j), then $\det(B) = -\det(A)$.
- if B is obtained from A by the row operation $(R_i \leftarrow kR_j + R_i)$ (replace row i by row i plus k times row j), then $\det(B) = \det(A)$.

The determinant also has the properties

- $\det(A^T) = \det(A)$.
- $\det(AB) = \det(A)\det(B)$.

A linear algebra course would cover all this in detail, but for now we do not care very much how it is computed, since we have matlab, and it will compute for us, both numerically and symbolically. Instead, we will explore an interpretation of the determinant which was introduced by Leibniz.

One of the major applications of determinants is in the establishment of a condition for the existence of a nontrivial solution for a set of linear

homogeneous algebraic equation. Suppose we have three unknowns x_1, x_2, x_3 (or more generally n equations with n unknowns)

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

$$b_1x_1 + b_2x_2 + b_3x_3 = 0$$

$$c_1x_1 + c_2x_2 + c_3x_3 = 0$$

The problem is to determine under what conditions there is any solution, apart from the trivial one $x_1 = 0, x_2 = 0, x_3 = 0$. If we use vector notation $\mathbf{x} = (x_1, x_2, x_3)$ for the solution and three rows $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3), \mathbf{c} = (c_1, c_2, c_3)$ of coefficients, then the above three equations become

$$\mathbf{a} \cdot \mathbf{x} = 0, \quad \mathbf{b} \cdot \mathbf{x} = 0, \quad \mathbf{c} \cdot \mathbf{x} = 0.$$

These three vector equations have the obvious geometrical interpretation that \mathbf{x} is orthogonal (or perpendicular) to \mathbf{a}, \mathbf{b} and \mathbf{c} . The important geometric interpretation of the determinant is the following. Let V denote the volume of the “distorted” cube spanned by $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Then V is equal to the absolute value of a determinant:

$$V = \left| \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \right|.$$

If the volume spanned by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ not zero, then clearly there is only the trivial solution $\mathbf{x} = \mathbf{0}$. Conversely, if the volume is zero, then $\mathbf{a}, \mathbf{b}, \mathbf{c}$ must lie in some plane, and \mathbf{x} can be any vector that is perpendicular to such a plane.

We show exercise 4 in matlab here (this is not a solution, just a visual effect to impress you and introduce some matlab plot commands).

$$\{(x_1, x_2) : (x_1, x_2) \cdot (v_1, v_2) = a\}.$$

Here “:” read “such that”. The whole line read we have a collection of points (x_1, x_2) such that the dot product of \mathbf{x} and \mathbf{v} is equal to a . You can think of \mathbf{x} and \mathbf{v} as two points or two vectors from origin $(0, 0)$ to that points alternatively. Now input in matlab: Let

```
v1 = 2, v2 = 1
plot(v1,v2,'o','markerfacecolor','y','markersize',8);
hold on; plot([0 v1],[0 v2]);
```

```
syms x1 x2
a=1.5;
x=[x1 x2]; v=[v1;v2];

x2=solve(x*v-a,x2)

x1=0:.1:1.5
x2=eval(x1)
plot(x1,x2,'o')
axis equal
print -djpeg first
Then you can view it in netscape.
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