Applied Math 9

Computation Handout 1

The name Matlab stands for *matrix laboratory*. Matlab is an interactive system whose basic data element is an array that does not require dimensioning. As a high-performance language for technical computing, it is much more efficient than C++/C, Fortran or Java. The trade off of high efficiency in writing is the overhead in running. That is, more work is shifted from manpower to machine. However, with the fast development of computer, the overhead has become less and less important. In addition Matlab integrates computation, visualization and symbolic deduction in an easy-to-use environment where problems and solutions are expressed in familiar mathematical notation. We now introduce some basic data elements and their operation.

1. *Vector*: Some of the things we measure are completely determined by their magnitudes. To record mass, length, or time, for example, we need only write down a number and name an appropriate unit of measure. But we need more than that to describe a force, displacement, or velocity, for these quantities have direction as well as magnitude. To describe a force, we need to record the direction in which it acts as well as how large it is. For example we use three numbers to describe a force

$$\mathbf{f} = [f_1\ f_2\ f_3] = [1\ 2\ 2.5]$$

and we call it a vector. The numbers 1, 2, 2.5 are the three components of the force in x, y, z directions. The magnitude of the vector is defined as

$$|\mathbf{f}| = \sqrt{f_1^2 + f_2^2 + f_3^2} = \sqrt{1^2 + 2^2 + 2.5^2} = 3.354$$

and it is an unique value. So the force is completely described by these three numbers. Similarly, to describe a body's displacement from a fixed reference point, we have to say in what direction it moved as well as how far it moved. We also use an array of three numbers, say

$$\mathbf{d} = [d_1\ d_2\ d_3\ ] = [2\ 1\ 1].$$

In general we name an array of data as *vector* and usually use bold letter to denote it. The size of the array (how many elements in the array) is called dimension of the vector and it can be very large. For an n dimensional vector $\mathbf{v}$, the magnitude(or norm) is usually defined as

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

2. *Dot products*: In a one dimensional setting, the work done by a constant force in displacing a mass is computed as the product of force and displacement. How do we compute work in a multi-dimensional setting, and in particular how is it computed if the vectors do not point in the same direction? Letting W denote the work done, you will find the answer in any high school textbook as

$$W = \mathbf{f} \cdot \mathbf{d} = f_1 d_1 + f_2 d_2 + f_3 d_3 = 1 \times 2 + 2 \times 1 + 2.5 \times 1 = 6.5.$$

Thus we define the *dot product* between two vectors as element-wise multiplication and summation. This definition has turned out to be very useful in many situations, and is also called the *scalar product* in some books. It is very easy to show that the commutative law

$$\mathbf{f} \cdot \mathbf{d} = \mathbf{d} \cdot \mathbf{f},$$

and distributive law

$$\mathbf{f} \cdot (\mathbf{d} + \mathbf{c}) = \mathbf{f} \cdot \mathbf{d} + \mathbf{f} \cdot \mathbf{c}$$

hold.

3. *Matrix*: A rectangular array of numbers like

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \end{pmatrix}$$

is called a *matrix*. We call A an "2 by 3" matrix because it has two rows and three columns. An "m by n" matrix has m rows and n columns, and the entry or element (number) in the ith row and jth column is often denoted by a_{ij}. Items are always given in the order "row" then "column". The matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \end{pmatrix}$$

has

$$a_{11} = 2, \quad a_{12} = 1, \quad a_{13} = 3$$

$$a_{21} = 1, \quad a_{22} = 0, \quad a_{23} = -2.$$

A matrix with the same number of rows as columns is a *square matrix* With each square matrix we associate a number det A, called the *determinant* of A, calculated from the entries of A. In matlab, it can be obtained by typing *det(A)*. An $n \times 1$ dimensional matrix is an n dimensional *column vector*, and an $1 \times n$ dimensional matrix is an n dimensional *row vector*.

4. *Matrix multiplication*: There are actually many definitions of "matrix multiplication", but we will state and use only the most common (and useful) one. Our definition of matrix multiplication can be based on the vector dot product. Consider an $n \times m$ matrix A and an $m \times r$ matrix B. We can think of A as a collection of n m-dimensional row vectors

$$A = \begin{pmatrix} \leftarrow & \mathbf{a}_1 & \rightarrow \\ \leftarrow & \mathbf{a}_2 & \rightarrow \\ \vdots & \vdots & \vdots \\ \leftarrow & \mathbf{a}_n & \rightarrow \end{pmatrix}.$$

Likewise, we can think of B as a collection of r m-dimensional column vectors

$$B = \begin{pmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_r \\ \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix}.$$

The product $C = AB$ (note the order!) is defined as the $n \times r$ matrix whose ijth entry is $\mathbf{a}_i \cdot \mathbf{b}_j$:

$$C = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \cdots & \mathbf{a}_1 \cdot \mathbf{b}_r \\ \vdots & \ddots & \vdots \\ \mathbf{a}_n \cdot \mathbf{b}_1 & \cdots & \mathbf{a}_n \cdot \mathbf{b}_r \end{pmatrix}.$$

Note that the dimensions of the row vectors and column vectors must be the same, so that while AB makes sense, BA does not (and we cannot define the product) unless $n = r$.

The products of matrices and vectors are defined by thinking of the row or column vector as a matrix. Thus if $\mathbf{f}$ is an m dimensional column vector then $A\mathbf{f}$ is well defined (column vectors multiply matrices on the *right*), while $\mathbf{f}A$ makes sense (multiplication by a vector on the left) only when $\mathbf{f}$ is an n dimensional row vector.

In terms of the specific examples given earlier in the handout, we have

$$A\mathbf{f} = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{f} \\ \mathbf{a}_2 \cdot \mathbf{f} \end{pmatrix} = \begin{pmatrix} (a_{11}f_1 + a_{12}f_2 + a_{13}f_3 \\ (a_{21}f_1 + a_{22}f_2 + a_{23}f_3 \end{pmatrix} = \begin{pmatrix} 11.5 \\ -4 \end{pmatrix}.$$

If you check the dimensions, you will find that the result of properly multiplying a vector (of either type) by a matrix is another vector of the same type and dimension. It is very important to keep in mind that the usual commutative relation does not hold. Thus, even if the dimensions are such that both AB and BA are well defined, in general

$$AB \neq BA.$$