Applied Math 9

Problem Set 1 for Markov Chains

1. Draw a diagram with 4 nodes and indicate the possible transitions and their respective probabilities, for the Markov chain with one step transition probabilities

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

2. Urn models are used in many applications, from statistical mechanics and communication theory to biology and computer science. Suppose that 2 black and 2 white balls are put into two urns in such as way that there are exactly 2 balls in each urn. Let the *state* of the Markov chain X_n identify the number of black and white balls in each urn. Identify all the states. Suppose the chain evolves in the following way. A ball will be chosen with equal probability from each urn (and independently between the two urns, and also independently of all past choices), and the two will be switched. Is this truly a Markov chain? If so, identify the 1-step transition matrix.

3. Consider the 1-step transition matrix

$$\begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix}$$

Suppose a Markov chain with this transition matrix starts off in states 1 and 2 with probability 1/2 for each. What is the probability that after one step the chain is in state 1? Let the initial distribution be 1/3 for state 1 and 2/3 for state 2. Now what is the probability that after one step the chain is in state 1? Does this chain have an invariant probability distribution? If so, what is it?

4. Let us suppose that in New England the quality of the apple crop varies as a Markov chain. There are three types of years: good (G), average (A), and poor (P). If this year is G, the (conditional) probabilities for next year are .4 (G), .4 (A), .2 (P). If this year is P, the probabilities for next year are .2 (G), .4 (A), .4 (P). If this year is A, the probabilities

for next year are .2 (G), .6 (A), .2 (P). Write down the transition matrix for this chain. Suppose this year is G. What is the conditional probability two years hence? If a farmer expects to be in business for a long time, he would be interested in the invariant probability distribution associated with this chain (we will discuss why in class). What is the invariant probability distribution?

5. A random variable Y is said to be uniform on the interval $[0, 1]$ if the probability that Y falls into the interval $[a, b]$, where $0 \le a \le b \le 1$, is exactly equal to the length of the interval: $(b - a)$. Suppose that you have a random number generator that could generate as many independent random variables that are uniform on $[0, 1]$ as you could want. Let's name them $Y_1, Y_2, Y_3, \ldots$. Suppose you are asked to *simulate* a finite state Markov chain with transition matrix P. This means that you must explain how you would construct a process X_n out of the Y's, so that X_n is a genuine Markov process with the desired transition matrix. Could you do it? To simplify, assume there are only 3 states, and that the chain starts in state 1 at time 0.