Handout 7 on Markov Chains: Pricing of Call and Put Options

In class we discussed the use of the so-called binomial tree model for the random fluctuations of an asset price. Below is a depiction of the transition probabilities for the binomial tree model when there are N time steps. The states are enumerated as indicated. For some $0 < p < 1$, we have

$$P\left\{X_{i+1} = c^m | X_i = c^k\right\} = \begin{cases} p & \text{if } m = k+1 \\ 1-p & \text{if } m = k-1 \\ 0 & \text{otherwise} \end{cases}$$

when $-N < k < N$. Note that the transition probabilities associated to states c^N and $1/c^N$ (numbered 1 and $2N+1$, respectively), are irrelevant, since we are only interested in what happens after starting in state 1, and if we start there then we never leave states c^N and $1/c^N$. (In fact it is only possible to get to them at time N.) To be concrete, however, we will take $p_{1,1} = p, p_{1,1} = (1-p)$ and $p_{2N+1,2N} = p, p_{2N+1,2N+1} = (1-p)$ (just as in the random walk on $2N+1$ states).

$$\begin{array}{ccccccccc}
 & & & & & & & & \text{State Number} \\
 & & & & & & & c^N & 1 \\
 & & & & & & & & 2 \\
 & & & & & c^2 & \cdots & & \vdots \\
 & & & & \nearrow & & & & \\
 & & & c & & & & & \\
 & & \nearrow & & \searrow & & & & \\
 & 1 & & & & 1 & \cdots & & \\
 & & \searrow & & \nearrow & & & & \\
 & & & \frac{1}{c} & & & & & \\
 & & & & \searrow & & & & \\
 & & & & & \frac{1}{c^2} & \cdots & & \\
 & & & & & & & \frac{1}{c^N} & 2N+1 \\
 \text{Time} & 0 & & 1 & & 2 & \cdots & N &
\end{array}$$

For example, if $N = 2$ then there are 5 states, and the transition matrix is

$$P = \begin{pmatrix} p & 1-p & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 1-p & 0 \\ 0 & 0 & p & 0 & 1-p \\ 0 & 0 & 0 & p & 1-p \end{pmatrix}.$$

The two step transition probabilities are

$$P^2 = \begin{pmatrix} p & p(1-p) & (1-p)^2 & 0 & 0 \\ p^2 & 2p(1-p) & 0 & (1-p)^2 & 0 \\ p^2 & 0 & 2p(1-p) & 0 & (1-p)^2 \\ 0 & p^2 & 0 & 2p(1-p) & (1-p)^2 \\ 0 & 0 & p^2 & p(1-p) & 1-p \end{pmatrix}.$$

In the European call option, we are granted the right (at time zero) to purchase a share of the stock at the price K at time N. If purchased, then we can immediately sell the asset at the going rate. Clearly we will only do this if the state of the price process exceeds the agreed upon price K (i.e., $X_N \geq K$). Thus the payoff vector, without taking inflation into account, is just

$$f = \begin{pmatrix} (c^N - K)^+ \\ (c^{N-1} - K)^+ \\ \vdots \\ \left(\frac{1}{c^{N-1}} - K\right)^+ \\ \left(\frac{1}{c^N} - K\right)^+ \end{pmatrix},$$

where a^+ equals a is $a \geq 0$, and 0 if $a \geq 0$. Suppose for example that $c \geq K \geq 1/c$, and $N = 2$. Then

$$f = \begin{pmatrix} (c^2 - K) \\ (c - K) \\ (1 - K) \\ 0 \\ 0 \end{pmatrix}.$$

If we do not account for inflation then the *expected payoff*, given that we start at state 1 at time zero (which is numbered as state 3), is

$$\begin{aligned} (P^2 f)_3 &= p^2 \cdot (c^2 - K) + 0 \cdot (c - K) + 2p(1-p) \cdot (1-K) + 0 \cdot 0 + (1-p)^2 \cdot 0 \\ &= p^2(c^2 - K) + 2p(1-p)(1-K). \end{aligned}$$

If, however, there is inflation with a rate R (so that a dollar tomorrow is worth $1/(1+R)$ dollars today), then the *expected discounted payoff* is

$$\frac{1}{(1+R)^2}(P^2 f)_3 = \frac{1}{(1+R)^2}\left[p^2(c^2-K)+2p(1-p)(1-K)\right].$$

For the general case with discounting, the expected discounted payoff (and hence fair price) is given by

$$\frac{1}{(1+R)^N}(P^N f)_{N+1}.$$

Example. If $K = 1.08, R = .04, c = 1.05, p = .5$ and $N = 2$, then $\frac{1}{(1+R)^2}(P^2 f)_3 = 0.0052$.

In the European put option, we are granted the right to *sell* an asset at a given price K. In this case we exercise the option only when the agreed upon price K is bigger then the going price X_N. Thus the only difference is that the payoff vector is now

$$f = \begin{pmatrix} (K-c^N)^+ \\ (K-c^{N-1})^+ \\ \vdots \\ \left(K-\frac{1}{c^{N-1}}\right)^+ \\ \left(K-\frac{1}{c^N}\right)^+ \end{pmatrix}.$$