Handout 7 on Markov Chains: Pricing of Call and Put Options

In class we discussed the use of the so-called binomial tree model for the random fluctuations of an asset price. Below is a depiction of the transition probabilities for the binomial tree model when there are $N$ time steps. The states are enumerated as indicated. For some $0 < p < 1$, we have

$$P\{X_{i+1} = c^m | X_i = c^k\} = \begin{cases} p & \text{if } m = k + 1 \\ 1 - p & \text{if } m = k - 1 \\ 0 & \text{otherwise} \end{cases}$$

when $-N < k < N$. Note that the transition probabilities associated to states $c^N$ and $1/c^N$ (numbered 1 and $2N + 1$, respectively), are irrelevant, since we are only interested in what happens after starting in state 1, and if we start there then we never leave states $c^N$ and $1/c^N$. (In fact it is only possible to get to them at time $N$.) To be concrete, however, we will take $p_{1,1} = p, p_{1,1} = (1 - p)$ and $p_{2N+1,2N} = p, p_{2N+1,2N+1} = (1 - p)$ (just as in the random walk on $2N + 1$ states).
For example, if $N = 2$ then there are 5 states, and the transition matrix is

$$P = \begin{pmatrix}
p & 1-p & 0 & 0 & 0 \\
p & 0 & 1-p & 0 & 0 \\
0 & p & 0 & 1-p & 0 \\
0 & 0 & p & 0 & 1-p \\
0 & 0 & 0 & p & 1-p \\
\end{pmatrix}.$$ 

The two step transition probabilities are

$$P^2 = \begin{pmatrix}
p & p(1-p) & (1-p)^2 & 0 & 0 \\
p^2 & 2p(1-p) & 0 & (1-p)^2 & 0 \\
p^2 & 0 & 2p(1-p) & 0 & (1-p)^2 \\
0 & p^2 & 0 & 2p(1-p) & (1-p)^2 \\
0 & 0 & p^2 & p(1-p) & 1-p \\
\end{pmatrix}.$$ 

In the European call option, we are granted the right (at time zero) to purchase a share of the stock at the price $K$ at time $N$. If purchased, then we can immediately sell the asset at the going rate. Clearly we will only do this if the state of the price process exceeds the agreed upon price $K$ (i.e., $X_N \geq K$). Thus the payoff vector, without taking inflation into account, is just

$$f = \begin{pmatrix}
(c^N - K)^+ \\
(c^{N-1} - K)^+ \\
\vdots \\
(\frac{1}{c^N} - K)^+ \\
(\frac{1}{c^{N-1}} - K)^+ \\
\end{pmatrix},$$

where $a^+$ equals $a \geq 0$, and 0 if $a \geq 0$. Suppose for example that $c \geq K \geq 1/c$, and $N = 2$. Then

$$f = \begin{pmatrix}
(c^2 - K) \\
(c - K) \\
(1 - K) \\
0 \\
0 \\
\end{pmatrix}.$$ 

If we do not account for inflation then the expected payoff, given that we start at state 1 at time zero (which is numbered as state 3), is

$$(P^2 f)_3 = p^2 \cdot (c^2 - K) + 0 \cdot (c - K) + 2p(1-p) \cdot (1 - K) + 0 \cdot 0 + (1-p)^2 \cdot 0$$

$$= p^2(c^2 - K) + 2p(1-p)(1-K).$$
If, however, there is inflation with a rate \( R \) (so that a dollar tomorrow is worth \( 1/(1 + R) \) dollars today), then the expected discounted payoff is

\[
\frac{1}{(1 + R)^2} (P^2 f)_2 = \frac{1}{(1 + R)^2} \left[ p^2 (c^2 - K) + 2p(1-p)(1-K) \right].
\]

For the general case with discounting, the expected discounted payoff (and hence fair price) is given by

\[
\frac{1}{(1 + R)^N} (P^N f)_{N+1}.
\]

**Example.** If \( K = 1.08, R = .04, c = 1.05, p = .5 \) and \( N = 2 \), then

\[
\frac{1}{(1+R)^2} (P^2 f)_2 = 0.0052.
\]

In the European put option, we are granted the right to sell an asset at a given price \( K \). In this case we exercise the option only when the agreed upon price \( K \) is bigger than the going price \( X_N \). Thus the only difference is that the payoff vector is now

\[
f = \begin{pmatrix}
(K - c^N)^+ \\
(K - c^{N-1})^+ \\
\vdots \\
(K - \frac{1}{c^N})^+ \\
(K - \frac{1}{c^N})^+
\end{pmatrix}.
\]