APPLIED MATH 9

Handout 5: Three Other Wave Equations, Their Normal Modes and Associated Frequencies

We will discuss three other wave equation problems.

VIBRATING STRING WITH DISSIPATION. Here the equation takes the form

$$u_{tt}(x,t) + \varepsilon u_t(x,t) = c^2 u_{xx}(x,t), 0 < x < L, t > 0,$$

and the boundary conditions

$$u(0,t) = u(L,t) = 0, t > 0.$$

The parameter $\varepsilon > 0$ reflects dissipative effects, and is typically small. If we look for solutions in the separated form $u(x,t) = f(x)g(t)$ we find that

$$\frac{g_{tt}(t) + \varepsilon g_t(t)}{g(t)} = c^2 \frac{f_{xx}(x)}{f(x)}.$$

Both sides must be a constant, which we call A. The boundary conditions again imply $f(0) = f(L) = 0$, and so we get

$$c^2 f_{xx}(x) - Af(x) = 0, f(0) = f(L) = 0$$

and

$$g_{tt}(t) + \varepsilon g_t(t) - Ag(t) = 0.$$

As before the first of these only has solutions of the form $\sin k\pi x/L, k = 1, 2, \ldots$, which occur when $A = -c^2 k^2 \pi^2/L^2$. This means the corresponding g equation is

$$g_{tt}(t) + \varepsilon g_t(t) + \frac{c^2 k^2 \pi^2}{L^2} g(t) = 0.$$

The general solution to this equation is

$$a_1 e^{-\frac{\varepsilon}{2}t} \sin \sqrt{\frac{c^2 k^2 \pi^2}{L^2} - \frac{\varepsilon^2}{4}}\, t + a_2 e^{-\frac{\varepsilon}{2}t} \cos \sqrt{\frac{c^2 k^2 \pi^2}{L^2} - \frac{\varepsilon^2}{4}}\, t,$$

where a_1 and a_2 are arbitrary constants. Notice that the temporal frequency is shifted slightly (if ε is small), and that the "simple" solutions to the wave equation

$$\sin \frac{k\pi x}{L} e^{-\frac{\varepsilon}{2}t} \sin \sqrt{\frac{c^2 k^2 \pi^2}{L^2} - \frac{\varepsilon^2}{4}}\, t, \sin \frac{k\pi x}{L} e^{-\frac{\varepsilon}{2}t} \cos \sqrt{\frac{c^2 k^2 \pi^2}{L^2} - \frac{\varepsilon^2}{4}}\, t$$

now tend to zero as $t \to \infty$.

WAVE EQUATION FOR A TUBE, WITH ONE END CLOSED AND ONE END OPEN. This is the model used for say a flute or a pipe. The "medium" here is the air in the tube, as opposed to the string in vibrating string example. The diameter of the tube is assumed small, so that the important variations occur along the tube's length, and thus the problem can be considered one-dimensional. The stopped end of the tube is considered to be at $x = 0$, and the open end at $x = L$. The function $u(x,t)$ represents air pressure now, and since we are interested in differences of pressure, we let $u(x,t)$ denote the difference between the pressure at position x at time t in the tube and the pressure in the room outside the tube. The correct boundary condition at the open end is then $u(L,t) = 0$, while the sealed end of the tube "disconnects" the air inside from that outside, and the correct boundary condition turns out to be $u_x(0,t) = 0$. The wave equation itself is the same as in the case of the vibrating string:

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), 0 < x < L, t > 0,$$

where c now reflects properties of air. If we look for solutions of the separated form $u(x,t) = f(x)g(t)$, we now get the equations

$$c^2 f_{xx}(x) - A f(x) = 0, f_x(0) = f(L) = 0$$

and

$$g_{tt}(t) - A g(t) = 0.$$

The solutions to the first equation are fundamentally different owing to the change from $f(0) = 0$ to $f_x(0) = 0$. In fact, instead of only sin's for solutions, the new boundary conditions permit only cos's for solutions. If we try $f(x) = \cos \lambda x$, then $f_x(0) = 0$ is automatic, while $f(0) = 0$ requires that λL equal one of

$$\frac{\pi}{2}, \frac{\pi}{2} + \pi, \frac{\pi}{2} + 2\pi, \dots.$$

Thus $\lambda = \pi/2L + k\pi/L, k = 0, 1, \dots$. Now from the equation the relation $\lambda^2 = -A/c^2$ must hold, and so the constant A must be of the form

$$A = -\left(\frac{\pi c}{2L} + k\frac{\pi c}{L} \right)^2, k = 0, 1, \dots.$$

Solving for the corresponding g part for a given k gives

$$a_1 \sin \left(\frac{\pi c}{2L} + k\frac{\pi c}{L} \right) t + a_2 \cos \left(\frac{\pi c}{2L} + k\frac{\pi c}{L} \right) t,$$

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and so we get simple solutions of the form

$$\cos\left(\frac{\pi}{2L}+\frac{k\pi}{L}\right)x\sin\left(\frac{\pi c}{2L}+k\frac{\pi c}{L}\right)t,\ \cos\left(\frac{\pi}{2L}+\frac{k\pi}{L}\right)x\cos\left(\frac{\pi c}{2L}+k\frac{\pi c}{L}\right)t.$$

The lowest frequency corresponds to $k=0$ and the normal mode $\cos(\pi x/2L)$. This frequency is $((\pi c/2L)/2\pi)=c/4L$. The first few frequencies are in fact

$$\frac{c}{4L},\frac{3c}{4L},\frac{5c}{4L},\dots\dots$$

These should be compared to the frequencies "allowed" by the boundary conditions that go with the vibrating string with no friction:

$$\frac{2c}{4L},\frac{4c}{4L},\frac{6c}{4L},\dots\dots$$

THE WAVE EQUATION FOR A DRUM HEAD. With c again representing properties of the physical medium, the wave equation for the drum head of radius R is

$$u_{tt}(x,y,t)=c^2\left(u_{xx}(x,y,t)+u_{yy}(x,y,t)\right),x^2+y^2<R,t>0.$$

The boundary condition is

$$u(x,y,t)=0,x^2+y^2=R,t>0.$$

Because of the geometry, it is easier to solve this equation in polar coordinates. You may remember that there is a "correction factor" that must be included when one changes coordinate systems, and the equation in polar coordinates (r,θ,t) is

$$u_{tt}(r,\theta,t)=c^2\left(u_{rr}(r,\theta,t)+\frac{1}{r}u_r(r,\theta,t)+\frac{1}{r^2}u_{\theta\theta}(r,\theta,t)\right),x^2+y^2<R,t>0,$$

and the boundary condition is

$$u(R,\theta,t)=0,0\le\theta<2\pi,t>0.$$

One can once again look for solutions in the separated form $u(r,\theta,t)=f(r)h(\theta)g(t)$. This produces a set of differential equations that require more advanced techniques than we have available right now. The solutions were presented in the form of animations on the web page

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http://www.gmi.edu/~drussell/Demos.html.

The allowed frequencies here are determined by *two* parameters, reflecting the two-dimensional nature of the drum head. The frequencies $\omega_{k,n}, k = 1, 2, \ldots, n = 1, 2, \ldots$, are implicitly described by the equations

$$J_k(\omega_{k,n} R/c) = 0,$$

where the different values of n correspond to different roots of the *Bessel function J_k*, which is itself a solution to what is called the *Bessel equation of order k*. The frequencies higher than the fundamental are no longer related in a simple fashion to the fundamental, resulting in a tone that differs greatly from the clear tone of the vibrating string.