Applied Math 9

Handout 4: Assembling the Solution to the Wave Equation for a More General Initial Condition

In the previous handout we discussed how one can represent certain functions in terms of the functions

$$\sin(k\pi x/L), k = 1, 2, \ldots.$$

We are interested in these functions because when they are used for the initial condition of the wave equation, the solution has a simple form. For example, as discussed in Handout 2 (and as you can check for yourself), if

$$u(x,t) = \sin(k\pi x/L)\sin(kc\pi t/L),$$

then $u(x,t)$ satisfies the wave equation

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), 0 < x < L, t > 0,$$

and the boundary conditions

$$u(0,t) = u(L,t) = 0, t > 0.$$

Note that for this solution we have the initial conditions

$$u(x,0) = 0, u_t(x,0) = \frac{kc\pi}{L}\sin(k\pi x/L).$$

Likewise,

$$u(x,t) = \sin(k\pi x/L)\cos(kc\pi t/L)$$

satisfies the wave equation, boundary conditions, and initial conditions

$$u(x,0) = \sin(k\pi x/L), u_t(x,0) = 0.$$

While this gives us some solutions to the wave equation, these are not "typical" initial conditions. How can we solve for more realistic conditions, such as a plucked string, or a string that is struck with a hammer at time 0? The *linearity* of the wave equation is crucial at this point. By linear, what we mean is that if $u_1(x,t)$ and $u_2(x,t)$ are solutions to the wave equation and boundary conditions, then so is $a_1u_1(x,t) + a_2u_2(x,t)$, where a_1 and a_2 are any real valued constants. This is easy to check:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\left[a_1u_1(x,t) + a_2u_2(x,t)\right] &= a_1\frac{\partial^2}{\partial t^2}u_1(x,t) + a_2\frac{\partial^2}{\partial t^2}u_2(x,t) \\
&= a_1c^2\frac{\partial^2}{\partial x^2}u_1(x,t) + a_2c^2\frac{\partial^2}{\partial x^2}u_2(x,t) \\
&= c^2\frac{\partial^2}{\partial x^2}\left[a_1u_1(x,t) + a_2u_2(x,t)\right],
\end{aligned}$$

and

$$a_1u_1(0,t) + a_2u_2(0,t) = 0, a_1u_1(L,t) + a_2u_2(L,t) = 0$$

for $t > 0$. Thus, e.g.,

$$\sin(\pi x/L)\cos(c\pi t/L) + \sin(2\pi x/L)\cos(2c\pi t/L)$$

is a solution to the wave equation. We can take any finite sum, and in fact, under conditions which guarantee convergence, we can even take infinite sums.

Example. We recall the string plucked to height 1 at the center:

$$f(x) = \begin{cases} 2x/L & 0 \le x \le L/2 \\ 2 - 2x/L & L/2 \le x \le L. \end{cases}$$

In the last handout we discussed how this function can be represented in the form

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(k\pi x/L),$$

where

$$a_k = \frac{2}{\pi^2k^2}\sin(k\pi/2).$$

If we want to solve the wave equation with $u(x,0) = f(x), u_t(x,0) = 0$, then perhpas we should try to build it from the known solutions

$$\sin(k\pi x/L)\cos(kc\pi t/L), k = 1, 2, \ldots$$

(we don't bother with the solutions $\sin(k\pi x/L)\sin(kc\pi t/L)$ because $u_t(x,0) = 0$). The representation for $f(x)$ suggests

$$u(x,t) = \sum_{k=1}^{\infty}\left(\frac{2}{\pi^2k^2}\sin(k\pi/2)\right)\sin(k\pi x/L)\cos(kc\pi t/L),$$

and in fact this turns out to be the solution.

Note that the frequency of $\cos(kc\pi t/L)$ (number of oscillations per second) is $kc/2L$. The frequencies of all parts of the solution are integer multiples of the lowest frequency $c/2L$ (also called the *fundamental*). The coefficients a_k measure the acoustic energy put into the different frequencies, and these numbers differ for different initial conditions. It is this different "weighting" of the frequencies that distinguishes the sound of the "plucked" string from that of say a string struck with a hammer. For the record, the solution to the wave equation when the string is struck in the middle by a hammer, which has zero displacement at time 0 but nonzero velocity, is

$$u(x,t) = \sum_{k=1}^{\infty} \left(\frac{1}{2L} \sin(k\pi/2) \right) \sin(k\pi x/L) \sin(kc\pi t/L).$$