Applied Math 9

Handout 3: Representing a Given Vector in Terms of a Fixed set of Orthonormal Vectors

It often turns out that one might want to describe an n-dimensional vector in terms of a fixed set of reference vectors. The most familiar instance is representation in terms of the standard basis vectors. If $\mathbf{v} = (v_1, \ldots, v_n)$, and if the standard basis vectors are denoted by $\mathbf{e}_i$, i.e.,

$$(\mathbf{e}_i)_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i, \end{cases}$$

then the representation is well-known and obvious:

$$\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n.$$

It may work out that the standard basis vectors are not so convenient. For example, a problem could involve the flight of an object such as a plane, and one may wish to consider reference vectors that reflect the orientation of the plane. In this handout we discuss how one can represent a given vector in terms of such a "nonstandard" set of basis vectors.

A set of vectors $\mathbf{w}_i, i = 1, \ldots, n$ are called *orthogonal* if

$$\mathbf{w}_i \cdot \mathbf{w}_j = 0$$

whenever $i \neq j$. They are called *orthonormal* if $\mathbf{w}_i \cdot \mathbf{w}_i = 1$ for $i = 1, \ldots, n$. We shall need one fact that is intuitive and natural, but which we do not prove.

A Basic Fact. Let $\mathbf{w}_i, i = 1, \ldots, n$ be an orthonormal set of vectors in n-dimensional space. If the vector $\mathbf{u}$ is perpendicular to each $\mathbf{w}_i, i = 1, \ldots, n$, then $\mathbf{u}$ must be the zero vector: $\mathbf{u} = \mathbf{0}$.

Now suppose that we want to represent a vector $\mathbf{v}$ in terms of these $\mathbf{w}_i, i = 1, \ldots, n$. We know that the "part" of $\mathbf{v}$ that points in the $\mathbf{w}_1$ is just

$$(\mathbf{v} \cdot \mathbf{w}_1)\mathbf{w}_1,$$

and there is no "part" of $\mathbf{w}_1$ that points in any of the directions $\mathbf{w}_i, i = 2, \ldots, n$. This suggests that if we add up the parts of $\mathbf{v}$ that point in each of the directions $\mathbf{w}_i, i = 1, \ldots, n$, then we should recover $\mathbf{v}$:

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{w}_1)\mathbf{w}_1 + \cdots + (\mathbf{v} \cdot \mathbf{w}_n)\mathbf{w}_n.$$

This is true, and in fact not hard to prove. Let

$$\mathbf{u} = \mathbf{v} - [(\mathbf{v} \cdot \mathbf{w}_1)\mathbf{w}_1 + \cdots + (\mathbf{v} \cdot \mathbf{w}_n)\mathbf{w}_n].$$

If we show that $\mathbf{u}$ is perpendicular to each $\mathbf{w}_i, i = 1, \ldots, n$, then $\mathbf{u}$ is zero, which means $\mathbf{v} = (\mathbf{v} \cdot \mathbf{w}_1)\mathbf{w}_1 + \cdots + (\mathbf{v} \cdot \mathbf{w}_n)\mathbf{w}_n$. We will use the properties of an orthonormal set of vectors. For any i, $\mathbf{w}_i \cdot \mathbf{w}_j = 0$ when $i \neq j$ and $\mathbf{w}_i \cdot \mathbf{w}_i = 1$ imply

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{w}_i &= \{\mathbf{v} - [(\mathbf{v} \cdot \mathbf{w}_1)\mathbf{w}_1 + \cdots + (\mathbf{v} \cdot \mathbf{w}_n)\mathbf{w}_n]\} \cdot \mathbf{w}_i \\
&= \mathbf{v} \cdot \mathbf{w}_i - [(\mathbf{v} \cdot \mathbf{w}_1)\mathbf{w}_1 \cdot \mathbf{w}_i + \cdots + (\mathbf{v} \cdot \mathbf{w}_n)\mathbf{w}_n \cdot \mathbf{w}_i] \\
&= \mathbf{v} \cdot \mathbf{w}_i - (\mathbf{v} \cdot \mathbf{w}_i)\mathbf{w}_i \cdot \mathbf{w}_i \\
&= \mathbf{v} \cdot \mathbf{w}_i - \mathbf{v} \cdot \mathbf{w}_i \\
&= 0.
\end{aligned}$$

Thus $\mathbf{u} = \mathbf{0}$.

It is easy to remember this formula. Just keep in mind that you *can* write $\mathbf{v}$ in the given form. If we suppose that

$$\mathbf{v} = a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n$$

for some (as yet) unknown constants $a_i, i = 1, \ldots, n$, then by dotting both sides with $\mathbf{w}_i$ we find

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{w}_i &= \{a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n\} \cdot \mathbf{w}_i \\
&= a_i(\mathbf{w}_i \cdot \mathbf{w}_i) \\
&= a_i,
\end{aligned}$$

so $a_i = \mathbf{v} \cdot \mathbf{w}_i$. This only works if the property $\mathbf{w}_i \cdot \mathbf{w}_i = 1$ holds. If this is not the case (the vectors are orthogonal but not *orthonormal*) then we need to take $a_i = \mathbf{v} \cdot \mathbf{w}_i / \mathbf{w}_i \cdot \mathbf{w}_i$.

Example. Suppose that

$$\begin{aligned}
\mathbf{w}_1 &= \frac{1}{\sqrt{3}}(1, -1, -1), \\
\mathbf{w}_2 &= \frac{1}{\sqrt{6}}(2, 1, 1), \\
\mathbf{w}_3 &= \frac{1}{\sqrt{2}}(0, 1, -1).
\end{aligned}$$

It is easy to check that these vectors are orthonormal. How can we represent $\mathbf{v} = (1, 0, 0)$? We compute

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{w}_1 &= \frac{1}{\sqrt{3}} \\
\mathbf{v} \cdot \mathbf{w}_2 &= \frac{2}{\sqrt{6}} \\
\mathbf{v} \cdot \mathbf{w}_3 &= 0.
\end{aligned}$$

Thus we have

$$\mathbf{v} = \frac{1}{\sqrt{3}}\mathbf{w}_1 + \frac{2}{\sqrt{6}}\mathbf{w}_2 = \frac{1}{3}(1, -1, -1) + \frac{2}{6}(2, 1, 1) = (1, 0, 0).$$

What about $\mathbf{v} = (1, 1, 1)$? Here

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{w}_1 &= -\frac{1}{\sqrt{3}} \\
\mathbf{v} \cdot \mathbf{w}_2 &= \frac{4}{\sqrt{6}} \\
\mathbf{v} \cdot \mathbf{w}_3 &= 0,
\end{aligned}$$

and

$$\mathbf{v} = -\frac{1}{\sqrt{3}}\mathbf{w}_1 + \frac{4}{\sqrt{6}}\mathbf{w}_2 = -\frac{1}{3}(1, -1, -1) + \frac{4}{6}(2, 1, 1) = (1, 1, 1).$$

Now that basic idea we have just discussed extends when *vectors* are replaced by *functions* on the interval $[0, L]$, the *dot* product is replaced by the *inner* product, and we allow an *infinite* sum. In the case of the vibrating string we consider only functions that are tied down at 0 and L, and in this case the basis vectors are most commonly replaced by the set of functions

$$\sin(\pi x/L), \sin(2\pi x/L), \ldots.$$

The idea was suggested by one of the Bernoulli brothers, and developed by Fourier. One must be extra careful through, to make sure the set of functions with respect to which we are expanding is "big enough." For example, if we left out $\sin(4\pi x/L)$ then we could not represent $\sin(4\pi x/L)$ in terms of what remains, since they are all perpendicular to $\sin(4\pi x/L)$.

Let $f(x)$ be continuous and satisfy $f(0) = f(L) = 0$. Suppose that f has the representation

$$f(x) = a_1 \sin(\pi x/L) + a_2 \sin(2\pi x/L) + \cdots.$$

We will not be precise at all with regard to the sense in which the infinite sum converges. By taking the inner product of f with each sin function, we see that

$$\begin{aligned}
\langle f(x), \sin(k\pi x/L)\rangle &= \langle\, a_1 \sin(\pi x/L) + a_2 \sin(2\pi x/L) + \cdots, \sin(k\pi x/L)\rangle \\
&= a_k \langle \sin(k\pi x/L), \sin(k\pi x/L)\rangle \\
&= a_k \frac{1}{L} \int_0^L (\sin(k\pi x/L))^2 \, dx \\
&= a_k \frac{1}{L}\, 2L,
\end{aligned}$$

so that we need

$$\begin{aligned}
a_k &= \frac{1}{2} \langle f(x), \sin(k\pi x/L)\rangle \\
&= \frac{1}{2L} \int_0^L f(x) \sin(k\pi x/L) dx.
\end{aligned}$$

Example. We consider the string plucked to height 1 at the center:

$$f(x) \begin{cases} 2x/L & 0 \le x \le L/2 \\ 2 - 2x/L & L/2 \le x \le L. \end{cases}$$

Then by evaluating the resulting integral we find

$$\begin{aligned}
a_k &= \frac{1}{2L} \int_0^{L/2} \frac{2}{L} x \sin(k\pi x/L) dx + \frac{1}{2L} \int_0^{L/2} \frac{2}{L} (L - x) \sin(k\pi x/L) dx \\
&= \frac{2}{\pi^2 k^2} \sin(k\pi/2).
\end{aligned}$$