Applied Math 9

Handout 3 for Markov Chains: An Example

Consider a process X_n defined as follows. The state space is $\{1, 2, 3, 4, 5, 6\}$. Suppose $X_n = i$. Then we pick up an i-sided die, and roll it (the roll is of course independent of the past). Suppose the die is fair (all outcomes equally likely), and that Y_n denotes the outcome. We then set $X_{n+1} = X_n + Y_n$ if $X_n + Y_n \leq 6$, and $X_{n+1} = X_n + Y_n - 6$ otherwise $[(X_n + Y_n - 1) \bmod 6 \text{ plus } 1]$. The one step transition matrix is then

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/5 & 1/5 & 1/5 & 1/5 & 0 & 1/5 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}.$$

Suppose we start on state 1 at time zero. Then the initial distribution is

$$\mathbf{v} = (1, 0, 0, 0, 0, 0).$$

The distribution at time n is then $\mathbf{v}P^n$. Below we give the distribution for various values of n.

$$\begin{array}{lll} n = 1 & \mathbf{vP^n} & = & (0, 1, 0, 0, 0, 0) \\ n = 2 & \mathbf{vP^n} & = & (0, 0, 0.5, 0.5, 0, 0) \\ n = 3 & \mathbf{vP^n} & = & (0.125, 0.125, 0, 0.1667, 0.2917, 0.2917) \\ n = 5 & \mathbf{vP^n} & = & (0.0852, 0.2338, 0.1796, 0.2361, 0.1236, 0.1417) \\ n = 10 & \mathbf{vP^n} & = & (0.1087, 0.2155, 0.1614, 0.2157, 0.1358, 0.1630) \\ n = 15 & \mathbf{vP^n} & = & (0.1081, 0.2162, 0.1622, 0.2162, 0.1351, 0.1621) \\ n = 20 & \mathbf{vP^n} & = & (0.1081, 0.2162, 0.1622, 0.2162, 0.1351, 0.1622) \end{array}.$$

Now suppose that the die has i sides at time n, but that $i = X_{n-1} + X_n$. (We suppose that $X_{-1} = 0$ and $X_0 = 1$.) We can construct the process just as before, but it will no longer be Markov.