Applied Math 9

Handout 2 for Math of Music: Natural Modes of Vibration and Simple Solutions to the Wave Equation

In the last handout we derived the partial differential equation satisfied by a vibrating string. In this handout we discuss the simplest solutions to the wave equation, the so-called "natural modes." An interesting web site which has animations of the natural modes for the string and drum head is: http://www.gmi.edu/~drussell/Demos.html. In particular, one should look in the section titled **Vibrational Modes of Continuous Systems.**

The wave equation is usually written

$$u_{tt}(x,t) = c^2 u_{xx}(x,t),$$

where $c^2 = T/\rho$. Since the string has zero displacement at $x = 0$ and $x = L$, we assume that these *boundary conditions* hold at all times:

$$u(0,t) = u(L,t) = 0 \text{ for all } t \geq 0.$$

A solution to this equation is a function $u(x,t)$ defined for $0 \leq x \leq L$ and $t \geq 0$ that satisfies the boundary conditions and the differential equation for all $0 < x < L$ and $t \geq 0$.

We attack the problem of solving this equation in a general setting by first looking for certain especially simple solutions, which are called the natural modes. More precisely, we look for solutions in what is called the *separated form.* This means we try to find solutions of the form

$$u(x,t) = f(x)g(t).$$

Suppose that $f(x)g(t)$ is in fact a solution. What conditions does this impose on f and g? We assume that both functions have two continuous derivatives. If we compute the partial derivatives of such a function, and plug into the equation, we arrive at

$$f(x)g_{tt}(t) = c^2 f_{xx}(x)g(t),$$

and unless g is identically equal to zero, the boundary condition implies

$$f(0) = f(L) = 0.$$

Now the equation $f(x)g_{tt}(t) = c^2 f_{xx}(x)g(t)$ is more restrictive than it looks at first glance. Without worrying about dividing by zero, suppose we divide both sides by $f(x)g(t)$. The resulting equation is

$$\frac{g_{tt}(t)}{g(t)} = c^2 \frac{f_{xx}(x)}{f(x)}.$$

Now a function of t can be equal to a function of x only if both functions are actually (the same) constant! Let us suppose that

$$\frac{g_{tt}(t)}{g(t)} = A, c^2 \frac{f_{xx}(x)}{f(x)} = A$$

for some number A. What does this tell us about f and g?

We look at g first. From the homework, you know that $\sin \lambda x$ is always a solution to

$$f_{xx}(x) + \lambda^2 f(x) = 0$$

and that the boundary conditions

$$f(0) = 0, f(L) = 0$$

restrict λ to be of the form $n\pi/L$ for a positive integer n. This means that if we want to solve

$$c^2 \frac{f_{xx}(x)}{f(x)} = A,$$

which we rewrite as

$$f_{xx}(x) - \frac{A}{c^2} f(x) = 0,$$

then a solution to this equation and the boundary conditions can be found when $-A/c^2 = n^2\pi^2/L^2$, or $A = -c^2n^2\pi^2/L^2$. Let us designate the corresponding solution by

$$f^{(n)}(x) = \sin \frac{n\pi}{L} x.$$

This tells us possible values for A. If we fix such a value, and consider the equation for g, we find

$$g_{tt}(t) - \frac{c^2n^2\pi^2}{L^2} g(t) = 0.$$

It is easy to check that

$$\sin \frac{cn\pi}{L} t \text{ and } \cos \frac{cn\pi}{L} t$$

are both solutions to this equation. In fact, as you will see in a differential equations course later on, the general solution to this equation is

$$g^{(n)}(t) = a \sin \frac{cn\pi}{L} t + b \cos \frac{cn\pi}{L} t,$$

where a and b are arbitrary constant.

The functions $f^{(n)}(x)$ are the natural modes of vibration of the string. The first few modes are

$$\sin \frac{\pi}{L} x, \sin \frac{2\pi}{L} x, \sin \frac{3\pi}{L} x.$$

They give the special shapes that are preserved by the wave equation if it starts off with that shape (eigenshapes, or eigenfunctions). The function $f^{(n)}(x)$ must be paired with the corresponding function $g^{(n)}(t)$, which describes the *frequency* of oscillations in the time variable. For example, the solution

$$u(x, t) = \sin \frac{\pi}{L} x \cos \frac{c\pi}{L} t$$

starts off at time $t = 0$ with displacement $u(x, 0) = \sin \frac{\pi}{L} x$, velocity $u_t(x, 0) = 0$ (since the derivative of $\cos \frac{c\pi}{L} t$ at $t = 0$ is 0), and over 1 second oscillates approximately $(c\pi/L)/(2\pi) = c/2L$ times. This gives the lowest possible frequency of the string, with the others being the multiples $nc/2L$. The solution

$$u(x, t) = \sin \frac{\pi}{L} x \sin \frac{c\pi}{L} t$$

starts off at time $t = 0$ with displacement 0, velocity $u_t(x, 0) = \frac{c\pi}{L} \sin \frac{\pi}{L} x$ (since the derivative of $\sin \frac{c\pi}{L} t$ at $t = 0$ is $\frac{c\pi}{L}$), and oscillates at the same frequency.