Handout 1: Derivation of the Equation of a Vibrating String

In this handout we derive (at least formally) the *partial differential equation* that is satisfied by a vibrating string. There are several assumptions that underlie the derivation. The chief among these are that only *small displacements* occur in the vertical direction, and that the displacements in the horizontal direction are even smaller, and in fact negligible with respect to the vertical displacements. The string is held at $x = 0$ and $x = L$, where $L > 0$ is a constant, and thus the displacement at these points is 0. The main notations are as follows.

$$
\begin{aligned}
u(x,t) &= \text{vertical displacement of the string at position } x \text{ at time } t \\
\theta(x,t) &= \text{the angle of the element of the string between } x \\
&\quad \text{and } x + \Delta x \text{ with respect to the horizontal} \\
T(x,t) &= \text{tension in the string at position } x \text{ and time } t \\
\rho(x) &= \text{mass density of the string at } x.
\end{aligned}
$$

The derivation of the equation is based on Newton's law that relates force and acceleration. This law states that the vector sum of all forces that act on a body equals the mass times the (vector) acceleration. If we consider the element of the string between x and $x + \Delta x$, then the following forces act on the element.

- The tension pulling to the right, which has a magnitude of $T(x+\Delta x, t)$, and acts at the angle of $\theta(x + \Delta x, t)$ above the horizontal.

- The tension pulling to the left, which has a magnitude of $T(x,t)$, and acts at the angle of $\theta(x,t)$ above the horizontal.

- Other external forces, such as gravity.

We will assume that the effects of the other forces are negligible with respect to the effect of tension.

Now the mass of the string element is mass density times the length, or approximately

$$\rho(x) \left[\Delta x^2 + \left(u(x + \Delta x, t) - u(x, t)^2 \right) \right]^{\frac{1}{2}}.$$

Using that Newton's law is phrased in terms of vectors (and in this particular case vectors with two components), we find from the vertical component that

$$\rho(x) \left[\Delta x^2 + \left(u(x + \Delta x, t) - u(x, t)^2 \right) \right]^{\frac{1}{2}} u_{tt}(x, t)$$
$$= T(x + \Delta x, t) \sin \theta(x + \Delta x, t) - T(x, t) \sin \theta(x, t).$$

If we divide by Δx and then send $\Delta x \to 0$, then the resulting equation is

$$\rho(x) \left[1 + (u_x(x, t))^2 \right]^{\frac{1}{2}} u_{tt}(x, t)$$
$$= \frac{\partial}{\partial x} [T(x, t) \sin \theta(x, t)]$$
$$= T_x(x, t) \sin \theta(x, t) + T(x, t) \cos \theta(x, t) \cdot \theta_x(x, t)$$

Now from the figure we see that

$$\tan \theta(x, t) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, t) - u(x, t)}{\Delta x} = u_x(x, t).$$

Using standard trigonometric identities, this implies

$$\sin \theta(x, t) = \frac{u_x(x, t)}{\left[1 + (u_x(x, t))^2 \right]^{\frac{1}{2}}}$$

$$\cos \theta(x, t) = \frac{1}{\left[1 + (u_x(x, t))^2 \right]^{\frac{1}{2}}}$$

$$\theta(x, t) = \tan^{-1} u_x(x, t)$$

$$\theta_x(x, t) = \frac{u_{xx}(x, t)}{\left[1 + (u_x(x, t))^2 \right]}.$$

2

Small displacement means that $\theta(x,t)$ is small in absolute value (i.e., close to zero), and hence so is $u_x(x,t)$. This simplifies much of the above. Indeed, we have (approximately)

$$
\begin{aligned}
\tan \theta(x,t) &\approx \theta(x,t) \\
\sin \theta(x,t) &\approx \theta(x,t) \\
\cos \theta(x,t) &\approx 1 \\
\theta_x(x,t) &\approx u_{xx}(x,t).
\end{aligned}
$$

Plugging these equations into

$$
\rho(x)\left[1 + (u_x(x,t))^2\right]^{\frac{1}{2}} u_{tt}(x,t) = T_x(x,t)\sin\theta(x,t) + T(x,t)\cos\theta(x,t)\cdot\theta_x(x,t)
$$

gives

$$
\rho(x)u_{tt}(x,t) = T_x(x,t)u_x(x,t) + T(x,t)u_{xx}(x,t).
$$

Next we use the horizontal component. Here Newton's law tells us that the horizontal force is approximately zero. Thus

$$
T(x + \Delta x, t)\cos\theta(x + \Delta x, t) - T(x,t)\cos\theta(x,t) = 0.
$$

If we divide by Δx and then send $\Delta x \to 0$, then the resulting equation is

$$
\frac{\partial}{\partial x}\left[T(x,t)\cos\theta(x,t)\right] = T_x(x,t)\cos\theta(x,t) + T(x,t)\sin\theta(x,t)\cdot\theta_x(x,t) = 0.
$$

For small amplitude vibrations, $\cos\theta(x,t)$ is close to one, $\sin\theta(x,t)$ is close to zero, and hence $T_x(x,t)$ is close to zero. In other words, T is a function of t only, and is determined by how hard you pull on the ends of the string.

Thus the equation

$$
\rho(x)u_{tt}(x,t) = T_x(x,t)u_x(x,t) + T(x,t)u_{xx}(x,t)
$$

simplifies to

$$
\rho(x)u_{tt}(x,t) = T(t)u_{xx}(x,t).
$$

If T is actually a constant (independent of t) and $\rho(x)$ is a constant (independent of x), then we get the simplified equation

$$
\rho u_{tt}(x,t) = T u_{xx}(x,t).
$$

This is usually written

$$
u_{tt}(x,t) = c^2 u_{xx}(x,t),
$$

where $c^2 = T/\rho$.