Problem Set 4 for Zero Sum Games

1. *Different ways to describe a plane.* In class we mentioned that there are two ways to describe a line. If the line is not vertical one can use its descriptions in terms of a function

\[ x_2 = mx_1 + b \]

with one independent variable \( x_1 \). In all cases we can describe it as a set of points

\[ \{(x_1, x_2) : (x_1, x_2) \cdot (v_1, v_2) = a\}, \]

where \( \mathbf{v} = (v_1, v_2) \) is perpendicular to the line.

The same holds true for planes in \( n \) dimensional space. Consider the plane in three dimensions described by the function

\[ x_3 = 4x_1 + 5x_2 + 2. \]

Describe this as a set of points of the form

\[ \{(x_1, x_2, x_3) : (x_1, x_2, x_3) \cdot (v_1, v_2, v_3) = a\}. \]

In other words, find \( \mathbf{v} \) and \( a \). Next suppose a plane is given as the set of points

\[ \{(x_1, x_2, x_3) : (x_1, x_2, x_3) \cdot (1, -2, 2) = -4\}. \]

Describe this plane also by a function with \( x_1 \) and \( x_2 \) the independent variables, and \( x_3 \) the dependent variable.

2. A convenient way to describe lines in dimension higher than 2 is also as a set of points. The set

\[ \{(x_1, x_2, x_3) : (x_1, x_2, x_3) = (v_1, v_2, v_3)t + (z_1, z_2, z_3) \text{ for some real number } t\} \]

is the line through \((z_1, z_2, z_3)\) that points in the direction \((v_1, v_2, v_3)\).

(We can also consider the line as a function of the independent variable \( t \) and with the dependent variables \( x_1, x_2, \) and \( x_3 \).) Analogous descriptions can be used for higher dimensions. Consider the planes

\[ \{(x_1, x_2, x_3) : (x_1, x_2, x_3) \cdot (1, -2, 2) = -4\} \]

\[ \{(x_1, x_2, x_3) : (x_1, x_2, x_3) \cdot (1, -1, 0) = 2\}. \]
Show that the line
\[(x_1, x_2, x_3) : (x_1, x_2, x_3) = (4, 4, 2)t + (0, -2, 0) \text{ for some real number } t\]
is in the intersection of the two planes.

3. A fundamental concept is that of linear independence. A set
\[
\{\mathbf{v}_j, j = 1, \ldots, J\}
\]
of \(n\)-dimensional vectors is linearly dependent when there are real numbers \(\{a_j, j = 1, \ldots, J\}\), not all of which are zero, such that \(\sum_{j=1}^{J} a_j \mathbf{v}_j = 0\). When a set of vectors are linearly dependent, one of the vectors can be written as a sum of the other vectors times appropriate constants. If vectors are not linearly dependent, then they are linearly independent. Consider the vectors
\[
\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.
\]

We claim that these vectors are linearly independent. To show this, you must argue that the only way \(a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 = 0\) can happen is if \(a_1 = a_2 = a_3 = 0\). Rewrite the equation \(a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 = 0\) in vector-matrix notation. Using what you know about determinants, show that they are linearly independent. What can you say about the sets of vectors
\[
\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]
and
\[
\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}?
\]

4. Consider the set of vectors
\[
\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\]
Are these vectors linearly dependent? What about any three from this set? Are they linearly dependent?
5. In the last homework you showed that the dot product defines a linear mapping from $R^n$ to $R$. Let $A$ be an $n \times n$ matrix. Then $Ax$ defines a mapping from $R^n$ to $R^n$. Is this mapping linear? In other words, if $x$ and $y$ are $n$ dimensional vectors, and if $a$ and $b$ are numbers, is

$$A(ax+by) = aAx + bAy?$$

6. Let $\theta$ be a number between 0 and $2\pi$. The matrix

$$
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
$$

defines a mapping of the plane to itself. Describe this mapping. Describe also the mapping defined by

$$
\begin{pmatrix}
2 & 0 \\
0 & 3
\end{pmatrix}.
$$

How might you describe the map defined by

$$
\begin{pmatrix}
0 & -3 \\
2 & 0
\end{pmatrix} = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
2 & 0 \\
0 & 3
\end{pmatrix}?