APPLIED MATH 9

Problem Set 4 for Zero Sum Games

1. *Different ways to describe a plane.* In class we mentioned that there are two ways to describe a line. If the line is not vertical one can use its descriptions in terms of a function

$$x_2 = mx_1 + b$$

with one independent variable x_1. In all cases we can describe it as a set of points

$$\{(x_1, x_2) : (x_1, x_2) \cdot (v_1, v_2) = a\},$$

where $\mathbf{v} = (v_1, v_2)$ is perpendicular to the line.

The same holds true for planes in n dimensional space. Consider the plane in three dimensions described by the function

$$x_3 = 4x_1 + 5x_2 + 2.$$

Describe this as a set of points of the form

$$\{(x_1, x_2, x_3) : (x_1, x_2, x_3) \cdot (v_1, v_2, v_3) = a\}.$$

In other words, find $\mathbf{v}$ and a. Next suppose a plane is given as the set of points

$$\{(x_1, x_2, x_3) : (x_1, x_2, x_3) \cdot (1, -2, 2) = -4\}.$$

Describe this plane also by a function with x_1 and x_2 the independent variables, and x_3 the dependent variable.

2. A convenient way to describe lines in dimension higher than 2 is also as a set of points. The set

$$\{(x_1, x_2, x_3) : (x_1, x_2, x_3) = (v_1, v_2, v_3)t + (z_1, z_2, z_3) \text{ for some real number } t\}$$

is the line through (z_1, z_2, z_3) that points in the direction (v_1, v_2, v_3). (We can also consider the line as a function of the independent variable t and with the dependent variables x_1, x_2, and x_3.) Analogous descriptions can be used for higher dimensions. Consider the planes

$$\{(x_1, x_2, x_3) : (x_1, x_2, x_3) \cdot (1, -2, 2) = -4\}$$
$$\{(x_1, x_2, x_3) : (x_1, x_2, x_3) \cdot (1, -1, 0) = 2\}.$$

Show that the line

$$\{(x_1, x_2, x_3) : (x_1, x_2, x_3) = (4, 4, 2)t + (0, -2, 0) \text{ for some real number } t\}$$

is in the intersection of the two planes.

3. A fundamental concept is that of *linear independence.* A set

$$\{\mathbf{v}_j, j = 1, ..., J\}$$

of n-dimensional vectors is linearly dependent when there are real numbers $\{a_j, j = 1, ..., J\}$, not all of which are zero, such that $\sum_{j=1}^{J} a_j \mathbf{v}_j = 0$. When a set of vectors are linearly dependent, one of the vectors can be written as a sum of the other vectors times appropriate constants. If vectors are not linearly dependent, then they are linearly independent. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

We claim that these vectors are linearly independent. To show this, you must argue that the only way $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = 0$ can happen is if $a_1 = a_2 = a_3 = 0$. Rewrite the equation $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = 0$ in vector-matrix notation. Using what you know about determinants, show that they are linearly independent. What can you say about the sets of vectors

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}?$$

4. Consider the set of vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Are these vectors linearly dependent? What about any three from this set? Are they linearly dependent?

5. In the last homework you showed that the dot product defines a linear mapping from R^n to R. Let A be an $n \times n$ matrix. Then $A\mathbf{x}$ defines a mapping from R^n to R^n. Is this mapping linear? In other words, if $\mathbf{x}$ and $\mathbf{y}$ are n dimensional vectors, and if a and b are numbers, is

$$A(a\mathbf{x}+b\mathbf{y}) = aA\mathbf{x}+bA\mathbf{y}?$$

6. Let θ be a number between 0 and 2π. The matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

defines a mapping of the plane to itself. Describe this mapping. Describe also the mapping defined by

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

How might you describe the map defined by

$$\begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}?$$