Applied Math 9

Problem Set 2 for Zero Sum Games

1. Consider a game with pure strategies and payoff matrix

$$\{a_{ij}, i = 1, \ldots, n, j = 1, \ldots, m\}.$$

Show that the lower game actually is lower than the upper game:

$$\max_{i=1,\ldots,n} \min_{j=1,\ldots,m} a_{ij} \leq \min_{j=1,\ldots,m} \max_{i=1,\ldots,n} a_{ij}.$$

Hint. Start by showing that for any i and j

$$\min_{j=1,\ldots,m} a_{ij} \leq a_{ij},$$

and continue until you build up to the desired inequality. Note that the quantity on the left above is a function of i, but not j. If you maximize both sides with respect to i, what happens?

2. Consider a 3×3 matrix A. If we consider A as given and x as a variable, then we know that the mapping from x to Ax defines a mapping from $I\!R^3 \to I\!R^3$. How much information do we need to determine A? Suppose we are told that

$$A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix}, A\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Does this entirely determine A? What if we are just given the first two equations above?

3. Two vectors $\mathbf{x}$ and $\mathbf{y}$ in $I\!R^n$ are said to be "perpendicular" if $\mathbf{x} \cdot \mathbf{y} = 0$. Consider two dimensions, and consider the line that contains $\mathbf{x}$ and the vector $\mathbf{0}$, and also the line that contains $\mathbf{y}$ and $\mathbf{0}$. Are these lines perpendicular in the usual sense?

4. *Lines and half spaces in $I\!R^2$.* Suppose that we are given a nonzero vector $= (v_1, v_2)$. Show that the set

$$\{(x_1, x_2) : (x_1, x_2) \cdot (v_1, v_2) = a\}$$

always describes a line in the x_1, x_2-plane that is perpendicular to $\mathbf{v}$. (The particular line depends of course on the value of a). We call

$$\{(x_1, x_2) : (x_1, x_2) \cdot (v_1, v_2) \leq a\}$$

the half space perpendicular to $\mathbf{v}$ with boundary line

$$\{(x_1, x_2) : (x_1, x_2) \cdot (v_1, v_2) = a\} .$$

This half space contains the point $\mathbf{0} = (0, 0)$ if $a \geq 0$, and does not contain it otherwise.

5. *Planes and half spaces in $\mathbb{R}^n$.* These formulas extend to the case of n-dimensional space, though the meaning of "perpendicular" when $n > 3$ is not so concrete. In terms of dot product and vector notation, how would you describe the plane that is perpendicular to $(1, 1, 4)$ and which contains the point $(0, 1, 0)$? How about the half space perpendicular to $(1, 2, 1, -2)$ which contains $(0, 0, 0, 0)$ and whose boundary plane contains the point $(1, -1, 0, 0)$?

6. Let $\mathbf{v}$ and $\mathbf{x}$ be vectors in $\mathbb{R}^n$. Think of the dot product $\mathbf{v} \cdot \mathbf{x}$ when $\mathbf{v}$ is known and $\mathbf{x}$ is considered as a variable as a mapping from $\mathbb{R}^n$ to $\mathbb{R}$. Show from the definition of dot product that this mapping is *linear,* which means that if $f(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$, then for any real numbers a_1 and a_2 and any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$,

$$f(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) = a_1 f(\mathbf{x}_1) + a_2 f(\mathbf{x}_2).$$

7. Consider the mapping from $\mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 + x_3 \\ x_1 + x_3 \\ x_2 + x_1 \end{pmatrix} .$$

Can you find a matrix A so that $T(\mathbf{x}) = A\mathbf{x}$?