APPLIED MATH 9

Problem set 1 for Zero Sum Games

Notation. Given an $n \times m$ matrix A with entries a_{ij}, the *transpose* of A, denoted by A^T (and sometimes by A') is the $m \times n$ matrix whose entry in the ith row and jth column is a_{ji}. E.g., if A is

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

then A^T is

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

1. We use $I\!R^n$ to denote n-dimensional Euclidean space (which we identify with the set of all n-dimensional real vectors). If $\mathbf{x} = [x_1, x_2, x_3]^T$ and

$$A = \begin{pmatrix} 4 & 2 & 9 \\ 3 & 5 & 4 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix},$$

then the mapping from $\mathbf{x}$ to $A\mathbf{x}$ is a mapping from $I\!R^3$ to $I\!R^4$ (often denoted $I\!R^3 \to I\!R^4$). What is the value of the map when $\mathbf{x} = [0, 1, 0]^T$?

2. Let

$$A = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

What is ABC? Is it equal to CBA? Does the order in which you do the computations matter (i.e., is $A(BC) = (AB)C$)?

3. • Write the following system of linear equations in the matrix form $A\mathbf{u} = \mathbf{b}$, where $\mathbf{u}$ is the column vector $[x\ y\ z]^T$.

$$2x - 3y + 4z = -19$$

$$6x + 4y - 2z = 8$$

$$x + 5y + 4z = 23.$$

- Solve the system using matlab.

4. - Let A be an arbitrary 3×3 matrix and let

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Show that $IA = A$ and also that $AI = A$. We call I the multiplicative identity matrix for all 3×3 matrices. The analogous formula defines the identity for $n \times n$ matrices.

   - Use $A = rand(3)$ to generate a "random" matrix and $I = eye(3)$ in matlab and then test your conclusion.

5. An $n \times n$ matrix A is called *symmetric* if $A = A^T$.

   - Show that for any 3×3 symmetric matrix A and any 3-dimensional row vector $\mathbf{x}$,

$$(\mathbf{x}A)^T = A\mathbf{x}^T.$$

   Hint: write A as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix},$$

   and $\mathbf{x} = [x_1, x_2, x_3]$.

6. Consider a game with an $n \times m$ payoff matrix A. Suppose we identify action i of Player 1 with the n-dimensional row vector $\mathbf{y}^{n,i} = \mathbf{x}$, where

$$\mathbf{x}_k = \begin{cases} 1 & \text{if} \quad k = i \\ 0 & \text{if} \quad k \neq i. \end{cases}$$

   Likewise we identify action j of Player 2 with the m-dimensional column vector $\mathbf{z}^{m,j} = \mathbf{x}$, where

$$\mathbf{x}_k = \begin{cases} 1 & \text{if} \quad k = j \\ 0 & \text{if} \quad k \neq j. \end{cases}$$

   Express the payoff when Player 1 chooses action i and Player 2 chooses action j in terms of the matrices $\mathbf{y}^{n,i}, \mathbf{z}^{m,j}$ and A. Hint: the outcome is an 1×1 matrix, i.e., a scalar. Check the dimensions.

7. Consider the payoff matrix

$$\begin{pmatrix} 3 & -3 & -2 & -4 \\ -4 & -2 & -1 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix}.$$

- Suppose that Player 1 has the advantage, and Player 2 must choose first. What is the best (minimax) choice for Player 2? If the roles are reversed, what is the best (maximin) choice for Player 1?
- Does the game have value? If so, what are the saddle point strategies.