APPLIED MATH 9

Handout 7 for Zero Sum Games: Linear Programming Formulation

In this handout we describe some of the properties of the new formulation of the lower game. Recall from Handout 6 that the value of the lower game can be described as follows.

$$\text{Maximize } x_{n+1}$$

over all points $(x_1, \ldots, x_n, x_{n+1})$ subject to the following constraints:

$$\sum_{i=1}^{n} a_{ij}x_i - x_{n+1} \geq 0$$

for $j = 1, \ldots, m$,

$$x_i \geq 0$$

for $i = 1, \ldots, n$, and

$$\sum_{i=1}^{n} x_i \geq 1$$

and

$$\sum_{i=1}^{n} x_i \leq 1.$$

The saddle point strategy for Player 1 is any given by the $(x_1, \ldots, x_n)$ part of any vector $(x_1, \ldots, x_n, x_{n+1})$ that achieves the maximum, with x_{n+1} of course the value of the lower game.

This problem is a special case of what is called *Linear Programming.* The standard form for a linear program varies from book to book, but we will use Matlab's form. A linear program is a minimization problem of the following form. We are given a k dimensional vector $\mathbf{c}$ (usually thought of as "costs") a $k \times r$ dimensional matrix B, and r dimensional vectors $\mathbf{b}$, and are asked to

$$\text{Minimize } \mathbf{c} \cdot \mathbf{x}$$

subject to the constraints

$$B\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0},$$

where $\mathbf{x} \geq \mathbf{0}$ means that the inequality holds component by component.

Notice that there are $r + k$ constraints, and that each one defines a half-space. It may be the the minimum is $-\infty$, or that there is no point that satisfies all the constraints. These are legitimate worries, but they do not occur with a well fomulated optimization problem, and we will not worry about them here.

In the game we do not know, a priori, the sign of x_{n+1}. This is accomodated by replacing x_{n+1} by two variables, x_{n+1} and x_{n+2}. The new x_{n+1} will equal the old x_{n+1} if it is non-negative, and 0 otherwise. Also, x_{n+2} will be the negative of the old x_{n+1} if that is non-positive, and 0 otherwise. Thus $x_{n+1} - x_{n+2}$ replaces the old x_{n+1}, and both x_{n+1} and x_{n+2} are greater than or equal to zero. With $\mathbf{x} = (x_1, \ldots, x_n, x_{n+1}, x_{n+2})$, we can now require $\mathbf{x} \geq \mathbf{0}$, where the inequality holds component by component.

We can convert the maximization problem that came from the lower game to a linear program as follows. We let $k = n + 2$, $r = m + 2$,

$$c_i = \begin{cases} 0 & \text{if } i = 1, \ldots, n \\ -1 & \text{if } i = n+1 \\ 1 & \text{if } i = n+2. \end{cases}$$

$$b_l = \begin{cases} 0 & \text{if } l = 1, \ldots, r-2 \\ -1 & \text{if } l = r-1 \\ 1 & \text{if } l = r. \end{cases}$$

and

$$B = \begin{pmatrix} -a_{11} & -a_{21} & \cdots & -a_{n1} & 1 & -1 \\ -a_{12} & -a_{22} & \cdots & -a_{n2} & 1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -a_{1m} & -a_{2m} & \cdots & -a_{nm} & 1 & -1 \\ -1 & -1 & \cdots & -1 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 \end{pmatrix}.$$

Then the value of the linear program is exactly the negative of the value of the lower game.

We should observe a few properties of linear programs that are useful. First, the linear program minimizes an affine function over a convex set. In the homework we gave a definition of an extreme point of a convex set. For the sets that come up in linear programming, these points, for obvious reasons, are called *feasible corner points.* Since a corner point is defined as the intersection of k planes and there are $r + k$ constraints, that means the number of feasible corner points is no bigger than the number of different combinations of $r+n$ equations taken k at a time. With $k = n+2$, $r = m+2$, that comes out to

$$\frac{(n+m+4)!}{(m+2)!(n+2)!},$$

a very big number indeed if n and m are large. Here are the properties:

PROPERTY 1. (i) If there is exactly one optimal solution, then it must be a corner point. (ii) If there are multiple optimal solutions, then there must be at least two corner points that are optimal.

PROPERTY 2. There are only a finite (though perhaps large) number of corner points.