Applied Math 9

Handout 6 for Zero Sum Games: Converting the Maxmin Problem to a Max Problem in the General Case

Consider again the lower game. We will let $x_i, i = 1, \ldots, n$ denote the probabilities used by Player 1, and $y_j, j = 1, \ldots, m$ those used by Player 2. Suppose the payoff matrix is A. According to its definition this game has the value

$$v_l = \max_{\sum_{i=1}^n x_i = 1, x_i \geq 0, i=1,\ldots,n} \; \min_{\sum_{i=1}^m y_j = 1, y_j \geq 0, j=1,\ldots,m} \sum_{i=1}^n \sum_{j=1}^m a_{ij} x_i y_j .$$

As in the simpler case looked at previously, a particular set of probabilities $\mathbf{x} = (x_1, \ldots, x_n)$ is optimal if and only if

$$\sum_{i=1}^n \sum_{j=1}^m a_{ij} x_i y_j \geq v_l \tag{1}$$

for every choice of the opponent's probabilites $\mathbf{y} = (y_1, \ldots, y_m)$. Note that we will also have equality for at least one choice of $\mathbf{y}$. This means that v_l can be described as follows: i*t is the biggest number such that (1) holds for all probabilities* $\mathbf{y}$. If we choose $\mathbf{y}$ to be 1 in the jth position and zero everywhere else, then

$$\sum_{i=1}^n a_{ij} x_i \geq v_l, \tag{2}$$

and so this inequality holds for all $j = 1, \ldots, m$. But if (2) holds, then since $\mathbf{y}$ is a probability vector (i.e., $\sum_{i=1}^m y_j = 1, y_j \geq 0, j = 1, \ldots, m$),

$$y_1 \sum_{i=1}^n a_{i1} x_i + \cdots + y_m \sum_{i=1}^n a_{im} x_i \geq y_1 v_l + \cdots + y_m v_l = v_l,$$

so that (1) holds. Thus (1) holds for all $\mathbf{y}$ if and only (2) holds for all $j = 1, \ldots, m$. We conclude that v_l *is the biggest number such that all the equations* (2) *hold for* $j = 1, \ldots, m$.

We therefore have another way to get at v_l throught a maximization problem (no max/min). Find the largest possible value of z, such that there are numbers $x_i, i = 1, \ldots, n$ so that all the following constraints holds.

$$\sum_{i=1}^n a_{ij} x_i \geq z, j = 1, \ldots, m$$

$$\sum_{i=1}^{n} x_i = 1, x_i \geq 0, i = 1, \ldots, n.$$

The variable z is often labeled x_{n+1}, and the problem rephrased in yet another way: Find the point $(x_1, \ldots, x_n, x_{n+1})$ with the largest value of x_{n+1}, that lies in the intersection of all the following half-spaces:

$$\sum_{i=1}^{n} a_{ij} x_i - x_{n+1} \geq 0$$

for $j = 1, \ldots, m$,

$$x_i \geq 0$$

for $i = 1, \ldots, n$, and

$$\sum_{i=1}^{n} x_i \geq 1$$

and

$$\sum_{i=1}^{n} x_i \leq 1.$$