Consider again the lower game. We will let \( x_i, i = 1, \ldots, n \) denote the probabilities used by Player 1, and \( y_j, j = 1, \ldots, m \) those used by Player 2. Suppose the payoff matrix is \( A \). According to its definition this game has the value

\[
v_l = \max_{\sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, \ldots, n} \min_{\sum_{j=1}^m y_j = 1, y_j \geq 0, j = 1, \ldots, m} \sum_{i=1}^n \sum_{j=1}^m a_{ij} x_i y_j.
\]

As in the simpler case looked at previously, a particular set of probabilities \( \mathbf{x} = (x_1, \ldots, x_n) \) is optimal if and only if

\[
\sum_{i=1}^n \sum_{j=1}^m a_{ij} x_i y_j \geq v_l
\]

for every choice of the opponent’s probabilities \( \mathbf{y} = (y_1, \ldots, y_m) \). Note that we will also have equality for at least one choice of \( \mathbf{y} \). This means that \( v_l \) can be described as follows: it is the biggest number such that (1) holds for all probabilities \( \mathbf{y} \). If we choose \( \mathbf{y} \) to be 1 in the \( j \)th position and zero everywhere else, then

\[
\sum_{i=1}^n a_{ij} x_i \geq v_l,
\]

and so this inequality holds for all \( j = 1, \ldots, m \). But if (2) holds, then since \( \mathbf{y} \) is a probability vector (i.e., \( \sum_{i=1}^m y_j = 1, y_j \geq 0, j = 1, \ldots, m \)),

\[
y_1 \sum_{i=1}^n a_{i1} x_i + \cdots + y_m \sum_{i=1}^n a_{im} x_i \geq y_1 v_l + \cdots + y_m v_l = v_l,
\]

so that (1) holds. Thus (1) holds for all \( \mathbf{y} \) if and only (2) holds for all \( j = 1, \ldots, m \). We conclude that \( v_l \) is the biggest number such that all the equations (2) hold for \( j = 1, \ldots, m \).

We therefore have another way to get at \( v_l \) through a maximization problem (no max/min). Find the largest possible value of \( z \), such that there are numbers \( x_i, i = 1, \ldots, n \) so that all the following constraints hold,

\[
\sum_{i=1}^n a_{ij} x_i \geq z, j = 1, \ldots, m
\]
\[\sum_{i=1}^{n} x_i = 1, x_i \geq 0, i = 1, \ldots, n.\]

The variable \(z\) is often labeled \(x_{n+1}\), and the problem rephrased in yet another way: Find the point \((x_1, \ldots, x_n, x_{n+1})\) with the largest value of \(x_{n+1}\), that lies in the intersection of all the following half-spaces:

\[
\sum_{i=1}^{n} a_{ij} x_i - x_{n+1} \geq 0
\]

for \(j = 1, \ldots, m,\)

\[x_i \geq 0\]

for \(i = 1, \ldots, n,\) and

\[
\sum_{i=1}^{n} x_i \geq 1
\]

and

\[
\sum_{i=1}^{n} x_i \leq 1.
\]