Applied Math 9

Handout 4 for Zero Sum Games: Elementary Solution Methods for Randomized Strategies

Example (Holmes and Moriarty, from von Neumann & Morgenstern).

Holmes is fleeing Moriarty in a chase from London to Europe. Possible routes:

$$\begin{array}{cccc} \text{London} & \to & \text{Canterbury} & \text{train} \\ \text{Canterbury} & \to & \text{Dover} & \text{train} \\ \text{Dover} & \to & \text{Europe} & \text{trian} \\ \text{Canterbury} & \to & \text{Europe} & \text{boat} \end{array}.$$

If Holmes and Moriarty depart from the train on the same stop, Moriarty shoots Holmes (and wins). The boat is also more risky than the train, and if taken there is only a .5 chance that Holmes will get to Europe. If Holmes gets to Europe he escapes (and wins). Thus Holmes' choices are (1) get off at Canterbury and take the boat, (2) get off at Dover. Moriarty's choices are (1) get off at Canterbury, (2) get off at Dover. The payoff matrix is the survival probability, which is thus

Strategy		Moriarty 1	Moriarty 2
Holmes	1	0	.5
	2	1	0

The lower (maximin, Holmes chooses first) value of the game is 0. The upper (minimax) value is .5. Thus there is no saddle point. What is the value if randomized strategies are allowed? Let p be the probability that Holmes gets off at Canterbury, and r the probability that Moriarty gets off at Canterbury. The expected payoff is then

$$1(1-p)r + .5p(1-r) = r + .5p - 1.5pr.$$

Lower game. Here, Moriarty knows p before choosing r. We want to compute the maximum of the payoff subject to $r \in [0,1]$. In general one might do this by what are called Lagrange multipliers, but in this case we can do it just by inspection. The maximum will be at an end point, and the particular endpoint just depends on the slope (as a function of r). We end up with the following cases:

$$r^* = \begin{cases} 0 & p < 2/3 \\ \text{anything} & p = 2/3 \\ 1 & p > 2/3. \end{cases}$$

If Moriarty uses r^*, then Holmes will see a payoff

$$\begin{array}{cc} .5p & p \leq 2/3 \\ 1-p & p \geq 2/3 \end{array},$$

which is minimized with value $1/3$ at $p = 2/3$.

Upper game. Here, Holmes knows r before choosing p. We want to compute the maximum of the payoff subject to $p \in [0, 1]$. We end up with the following cases:

$$p^* = \begin{cases} 1 & r < 1/3 \\ \text{anything} & r = 1/3 \\ 0 & r > 1/3. \end{cases}$$

If Holmes uses p^*, then Moriarty will see a payoff

$$\begin{array}{cc} .5(1-r) & r \leq 1/3 \\ r & r \geq 1/3 \end{array},$$

which is minimized with value $1/3$ at $r = 1/3$.

Thus the saddle point strategies among randomized strategies are $p^* = 2/3$, $r^* = 1/3$.