Applied Math 9

Handout 3 for Zero Sum Games: Randomized Strategies

Before formulating games with randomized strategies, we review some very elementary definitions from probability. Suppose that the random variable X can take a value from among the finite set of real numbers $S = \{a_1, \ldots, a_n\}$. Let p_i denote the probability of outcome a_i: $p_i = P\{X = a_i\}$. We refer to $\{p_i, i = 1, \ldots, n\}$ as the *distribution* associated with X. The rules of probability require that $p_i \geq 0, i = 1, \ldots, n$, and that $\sum_{i=1}^{n} p_i = p_1 + \cdots + p_n$=1. We define the *expected value* of X to be

$$EX = \sum_{i=1}^{n} a_i p_i.$$

Now suppose that we have a collection of random variables $\{X_k, k = 1, ..., K\}$, all of which take on only the values $\{a_1, \ldots, a_n\}$. We say that these random variables *have the same distribution* as X if

$$P\{X = a_i\} = P\{X_k = a_i\} \text{ for all } i = 1, \ldots, n, k = 1, \ldots, K.$$

We will be very interested in understanding the degree and sense in which one random variable is related to other random variables. One very simple setting is where they are essentially unrelated. Consider a sequence of values from S, one for each random variable: $\{b_k \in S : k = 1, \ldots, K\}$. The collection $\{X_k, k = 1, ..., K\}$ is said to be *independent* if for any such sequence

$$P\{X_1 = b_1, X_2 = b_2, \ldots, X_K = b_K\} = P\{X_1 = b_1\}P\{X_2 = b_2\} \times \cdots \times P\{X_K = b_K\}.$$

If the collection $\{X_k, k = 1, ..., K\}$ all have the same distribution and are also independent, then it is called a collection of *independent and identically distributed* (iid) random variables.

We also note that one defines independence in the analogous way when each X_k can take values from a different (but still finite) set.

Example. Let $S = \{-1, 0, 1\}$. Suppose that X takes on more than one value with positive probability, and let $X_1 = X, X_2 = -X$. Then $\{X_1, X_2\}$ are *not* independent, and hence not iid. For example, if $P\{X_1 = -1\} = p \in (0, 1)$, then $P\{X_2 = 1\} = p$. But

$$p = P\{X = -1\} = P\{X_1 = -1, X_2 = 1\} \neq P\{X_1 = -1\}P\{X_2 = 1\} = p^2.$$

Randomized strategies. Suppose that each player will choose his or her action according to some pre-selected distribution. Thus Player 1 will have a probability distribution $\{x_i, i = 1, \ldots, n\}$, and Player 2 will have a probability distribution $\{y_j, j = 1, \ldots, m\}$. At this time we will assume that neither player has any information on the other player, and that the selection of actions of the two players are *independent*. Thus the probability that Player 1 chooses action i and Player 2 chooses action j is the product: $x_i y_j$. Let A be the payoff matrix. When considering randomized strategies it is customary for each player to optimize the *expected payoff*

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_i y_j.$$

We will explain why in a moment, after discussing the *law of large numbers*.

The law of large numbers. Suppose we return to the collection $\{X_k, k = 1, ..., K\}$ of iid random variables, but suppose that in fact we are dealing with an *infinite* collection $\{X_k, k = 1, ...\}$ (although such sequences exist, there are some mathematical subtleties which we will ignore). Suppose also that we keep track of the *sample average*, which is just the average of the actual outcomes. The sample average for the first K is thus

$$m_K = \frac{X_1 + X_2 + \cdots + X_K}{K}.$$

Then the law of large numbers states that with probability 1, m_K (the sample mean) converges to EX (the probabilistic mean) as K tends to ∞. If you think of X_k as the (random) outcome of an experiment, and if the experiments are "independent" in the sense defined previously, then the average of the experiments tends to the probabilistic average. Can you find a sequence of random variables which are *not* independent and for which the sample mean does *not* converge to the expected value?

We now return to the game model. Here, X_k would represent the random payoff. Suppose the game is played repeatedly with the same strategies, and that the actions of the players are *independent* from game to game. Then the average of the actual payoffs will converge to the expected payoff. It is this convergence which justifies the use.