Applied Math 9

Handout 2 for Zero Sum Games: Definitions related to pure strategies

In Handout 1 we described a few simple examples in Game Theory. In this handout we introduce some additional concepts and definitions. We first give the definitions, and then illustrate via the examples.

1. *Action:* Each player chooses from among a finite set of possible options, also called actions.

2. *Strategy:* A strategy is a rule for choosing an action. In general, a strategy will depend on the available information. For example, Player 1 may know the action chosen by Player 2 prior to selecting his or her own action. In this case, a strategy could be simply a mapping from the possible actions of Player 2 into the possible actions of Player 1. On the other hand, Player 1 may be forced to choose an action with no knowledge of Player 2's choice. We will refine the definition of a strategy later on.

3. *Pure strategy*: Remember that a strategy is a rule for choosing an action. If, given the available information, the rule dictates an explicit, particular action, then the strategy is called pure. The intention is to distinguish from *randomized strategies* which we will discuss in great detail later on.

4. *Lower and upper values of the pure game*: Suppose that Player 1 wants to maximize whilst Player 2 wishes to minimize. Suppose also that Player 2 gets to see the choice of Player 1 before selecting his own action, and that only pure strategies are allowed. Player 1 will evaluate the payoff for each possible action. For each fixed action of Player 1, this is equal to the minimum over all possible actions of Player 2. Player 1 will then maximize over the actions available to him. The resulting value is called the lower value (or the maximin value) of the pure game. Likewise, if the minimizing player (Player 2) is forced to choose first and Player 1 gets to see that choice before selecting, then the resulting value is called the upper value (or minimax value) of the pure game.

5. *Value of the pure game*: If the upper and lower values coincide, then the game is said to have value. In such a case, there is no "information advantage."

6. *Saddle point strategies:* A saddle point is an equilibrium point of the game. If the players use saddle point strategies, then neither player is motivated to unilaterally depart from his or her equilibrium decision. The notion of a saddle point also depends on the information that is made available to the two players. In the setting of pure strategies, saddle point strategies assume knowledge of the other player's choice of action. Players using saddle point strategies will not modify their actions. Not every game has a saddle point among pure strategies. In particular, the game must have value for saddle point strategies to exist.

7. *Solution to a game* This is simply a strategy for each player that gives the best possible payoff. Note that this depends on the information available to each player. A solution to the game thus consists of three things: an optimal strategy for player1, an optimal strategy for player 2 and the value of the game. Our purpose is to find the solution.

The table below illustrates the calculation of the upper (minimax) and lower (maximin) pure games for Example 2 of Handout 1.

Strategy		Player 2 $[1,1]$	$[2,1]$	$[1,2]$	$[2,2]$	Minimum
Player 1	$[1,1]$	0	-3	2	0	-3
	$[2,1]$	3	0	0	-4	-4
	$[1,2]$	-2	0	0	3	-2 $\leftarrow$ Maximin
	$[2,2]$	0	4	-3	0	-3
	Maximum	3	4	2	3	
				$\uparrow$ Minimax		

If Player 1 must choose first (the lower game, since Player 2 has the advantage), then we compute the minimum along each fixed row. This gives the best that Player 2 can do for each particular action of Player 1. Player 1 then maximizes among the minima, obtaining the maximin value of -2. Given this information structure, the optimal strategy for Player 1 is to choose $[1,2]$. The optimal strategy for Player 2 is a mapping that depends on the observed choice of Player 1: choose $[2,1]$ if $[1,1]$ is observed, $[2,2]$ if $[2,1]$, $[1,1]$ if $[1,2]$, and $[1,2]$ if $[2,2]$.

Likewise, the value of the upper game (Player 1 now has advantage) is 2. The pure game does not have value, and there is no saddle point strategy. Suppose Player 1 tentatively chooses $[1, 2]$, and reveals the choice. Then Player 2 will (tentatively) choose $[1, 1]$. If this choice is revealed to Player 1, then they will switch to $[2, 1]$, and so on.

If $\{a_{ij}, i = 1, \ldots, I, j = 1, \ldots J\}$ is the payoff matrix, then in general

$$\text{lower game} = \text{maximin} = \max_{i=1,\ldots,I} \min_{j=1,\ldots J} a_{ij}$$

and

$$\text{upper game} = \text{minimax} = \min_{j=1,\ldots J} \max_{i=1,\ldots,I} a_{ij}.$$

Note that the "stronger" player (the player with the "information advantage") appears on the inside in these expressions.

Next consider Example 3 of Handout 1. The calculation of the upper and lower values is below.

Strategy		Player 2: 1	2	3	Minimum
	1	-3	-2	6	-3
Player 1	2	2	0	2	0 ← Maximin
	3	5	-2	-4	-4
Maximum		5	0	6	
			↑ Minimax		

Note that the two values coincide, with value 0. Suppose that both players choose action 2. Then Player 1 will make no change, since any change leads to -2 (rather than 0). Likewise, Player 2 is not inclined to make a change of action. These are saddle point strategies. The saddle point is $(2, 2)$.