Applied Math 9

Handout 1 for Zero Sum Games: Examples.

For school children, a "game" is often thought of as a mere pastime, a way to spend their day avoiding homework, perhaps playing hide-and-seek. For adults, the term refer to chess or poker, in which each player is seeking strategy that will give them an advantage over their opponents. The latter situations, in which the outcome of the game is determined by the strategies employed by the players, form the starting point of what we now term the mathematical *theory of games*. What makes the game theory a "theory" rather than a collection of heuristics and rules of thumb is a formulation as a *linear programming* problem and as a mini-max problem, whose solution will identify the optimal strategies for the various players in the game. In addition to chess or poker, the term "game" here has an extended meaning. Many battles in World War II can be formulated as games and "solved" by the theory. Many financial and business situations can be also formulated as games, as can political conflicts, such as debates in the U.S. Senate. However, we will not consider all games, and in fact limit our attention to two-player zero-sum games. The mathematical reason for this is that two-player games can be solved completely when the interests of the two players are in direct opposition (a situation referred to as "zero-sum"). The practical reason is that many complex situations can be simplified and abstracted as two-player games. For example, although debates in the Senate involve more than 100 players (each with his own opinion), it can often be abstracted as a conflict between Democrats and Republicans. There is often a dominant conflict in complex situations that largely determines the outcome. As with any subject, a grasp of the basic concepts is absolutely essential, and elementary problems can go a long way in aiding understanding. We start our study of the game theory by looking some very simple and familiar games.

Example 1. *Scissors, Rock and Paper.* This toy problem introduces the basic ideas. There are two players for the game and the well known rules are:

- Paper covers Rock
- Rock smashes Scissors
- Scissors cut Paper

A matrix description of the game. A game is determined by the set of *actions* that the players can take and the corresponding *payoffs*. By tradition, the payoff is arranged as a *matrix*. The payoff can be denominated in many ways, and we often assume a payoff in money (dollars).

In the present problem, each player can choose one of Scissors, Rock or Paper (denoted S, R, and P, respectively). Assuming there is a \$1 payoff if Player 1 wins (and $-\$1$ if Player 2 wins). The payoff table is then

Strategy		Player 2 P	S	R
	P	0	-1	1
Player 1	S	1	0	-1
	R	-1	1	0

The *payoff matrix* associated with this game is then

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

By convention, the actions selected by Player 1 correspond to the *rows* of the matrix, while those of Player 2 correspond to the columns. The entries of A are usually denoted by a_{ij}, where a_{ij} refers to the entry in the ith row and jth column. Thus $a_{13} = 1$ and $a_{31} = -1$.

Example 2. *The Finger Game called Morra.* Morra is a hand game played for points by two people. Both players show either one or two fingers, and simultaneously call, out loud, the number of fingers that the other player will show. A correct call by one player but not the other wins the number of points showing as fingers (which can be either 2, 3, or 4). If both players call correctly or both call incorrectly there is no winner and no points are awarded.

Morra is thus a zero-sum two-person game. If by $[a,b]$ we designate that a player shows a fingers and calls that the other player will show b fingers we can construct the following table of points that will be won by Player 1:

Strategy		Player 2 $[1,1]$	$[2,1]$	$[1,2]$	$[2,2]$
	$[1,1]$	0	-3	2	0
Player 1	$[2,1]$	3	0	0	-4
	$[1,2]$	-2	0	0	3
	$[2,2]$	0	4	-3	0

The payoff matrix is thus

$$A = \begin{pmatrix} 0 & -3 & 2 & 0 \\ 3 & 0 & 0 & -4 \\ -2 & 2 & 0 & 3 \\ 0 & 4 & -3 & 0 \end{pmatrix}$$

Example 3. *A Game of Politicians.* Two politicians are running for U.S. president and the final 2 days are expected to be crucial because of the closeness of the race. Therefore, both politicians want to spend these days in two key states: Florida and California. Each player has the following three strategies:

- Strategy 1 = spend 1 day in each state.
- Strategy 2 = spend both days in Florida.
- Strategy 3 = spend both days in California.

Owing to existing polling data, research on the effectiveness of in-person campaigning, etc., the payoff matrix takes the form

$$\begin{pmatrix} -3 & -2 & 6 \\ 2 & 0 & 2 \\ 5 & -2 & -4 \end{pmatrix}.$$

How should the politicians invest their time? We will see later on that this game, in contrast to the previous two, has a simple solution no matter what information is shared between the two opponents regarding their intentions.

Example 4. *The Concord Arsenal Game.* On the evening of April 16, 1775 (three days prior to the American revolution), the minuteman Bigelow rode 17 miles from Boston to Concord and warned the patriots that the British had decided to attack the American arsenal at Concord. The colonists did not know which way the British had chosen to come-whether by land or by sea.

Suppose we were to re-create the problem as one of game theory. The American force is too small to defend both routes and they must choose to defend one or the other and take the consequences. Suppose that in fact the British are low on Ammunition, and if the two forces meet, the British will retreat. This scores for the Americans. If the forces do not meet, the British reach the Concord arsenal and increase their ammunition. If that happens both sides must have a plan of action for when the British return from Concord. The Americans can either lay in ambush on the known path of return, or to move in and attack the British at the arsenal. At the same time, the British can either leave the arsenal immediately by day or wait and withdraw by night. These various possibilities lead to four different courses of action that can be taken by each side labeled A1, A2, A3, A4, B1, B2, B3, B4. Let A denote Americans and B denote British.

- A1: defend by land, then ambush,
- A2: defend by land, then attack,
- A3: defend by sea, then ambush,
- A4: defend by sea, then attack.
- B1: go by sea, then leave immediately,
- B2: go by sea, then wait for night,
- B3: go by land, then leave immediately,
- B4: go by land, then wait for night.

Now let's consider the payoffs. The first point to note is that the interests of the two forces are directly opposed. Thus, the Concord Arsenal Game is a two-player, zero-sum game. Hence we only measure the payoffs of Americans, with British receiving the negative of this amount in each case.

Since the British are low on ammunition, the Americans have the advantage if the two forces meet when British go to the arsenal. This happens if the British take action B3 or B4 while Americans respond with A1 or A2. It also occurs if the British choose B1 or B2 and the Americans react with A3 or A4. Any of these four combinations yields the best possible outcome for the Americans and we assign 2 points payoff for the Americans in each such case.

Strategy		Player 2 (British)			
		$B1$	$B2$	$B3$	$B4$
Player 1 (Americans)	$A1$	2	0	2	2
	$A2$	0	1	2	2
	$A3$	2	2	2	0
	$A4$	2	2	0	1

For the remaining entries in the table (that is Americans missed British on their way to Concord), we argue as follow. If the British meet the ambush by day they will be destroyed (score 2 for the Americans). This happens with the pair of action (A1,B1) and (A3,B3). If the British meet the ambush by night they can filter through with some small loss (score 0). These are the pairs (A1,B2) and (A3,B4). If the Americans attack the arsenal and the British have already left, the choices (A2,B1) and (A4,B3), then the Americans score 0 points. But if the Americans attack and find the British waiting for night, both sides suffer heavy losses and the Americans score 1. These decisions correspond to the pairs (A2,B2) and (A4,B4). The payoff matrix thus is

$$A = \begin{pmatrix} 2 & 0 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 0 & 1 \end{pmatrix}$$

Example 5. *Prisoner's Dilemma game: A non-zero-sum game.* Another category of games is the non-zero-sum game, then the two opponents interests are not in total conflict. In such a case there may be cooperation. We will not discuss such games at length, but just give one example. In our (well-known) example the players are partners in a crime who have been captured by the police. Each suspect is placed in a separate cell, and offered the opportunity to confess to the crime. Since the game is not zero-sum, we will need two matrices representing the payoffs for the two players. The game can be represented by the following table, with the first entry in any column and row giving the payoff for Player 1, and the second that for Player 2:

Strategy		Player 2 Don't confess (D)	Player 2 confess (C)
Player 1	Don't confess (D)	5, 5	0, 10
	confess (C)	10, 0	1, 1

Higher numbers (which are sometimes referred to as "utility") are regarded as better by each player. If neither suspect confesses, they go free, and split the proceeds of their crime. This represents 5 units of utility for each suspect. However, if one prisoner confesses and the other does not, the prisoner who confesses testifies against the other in exchange for going free and gets the entire 10 units of utility, while the prisoner who did not confess goes to prison and gets nothing. If both prisoners confess, then both are given a reduced term, but both are convicted, which we represent by giving each 1 unit of utility: better than having the other prisoner confess, but not so good as going free. You may observe that it is not a zero-sum game (it is a cooperative game, not in direct conflict). However, this game has fascinated game theorists for a variety of reasons. First, it is a simple representation of a variety of important situations. For example, instead of confess/not confess we could label the strategies "contribute to the common good" or "behave selfishly." This captures a variety of situations that economists describe as public goods problems. An example is the construction of a bridge. It is best for everyone if the bridge is built, but best for each individual if someone else builds the bridge. This is sometimes referred to in economics as an externality. Similarly this game could describe the alternative of two firms competing in the same market, and instead of confess/not confess we could label the strategies "set a high price" and "set a low price." Naturally it is best for both firms if they both set high prices, but best for each individual firm to set a low price while the opposition sets a high price.

A second feature of this game is that it is clear how an intelligent individual should behave. No matter what a suspect believes his partner is going to do, it is always best to confess. If the partner in the other cell is does not confess, it is possible to get 10 instead of 5. If the partner in the other cell confesses, it is possible to get 1 instead of 0. Yet the pursuit of individually sensible behavior results in each player getting only 1 unit of utility, much less than the 5 units each that they would get if neither confessed. This conflict between the pursuit of individual goals and the common good (or "common bad" in the present example!) is at the heart of many game theoretic problems.

A third feature of this game is that it changes in a very significant way if the game is repeated, or if the players will interact with each other again in the future. Suppose for example that after this game is over, and the suspects either are freed or are released from jail they will commit another crime and the game will be played again. In this case in the first period the suspects may reason that they should not confess because if they do not their partner will not confess in the second game. Strictly speaking, this conclusion is not valid, since in the second game both suspects will confess no matter what happened in the first game. However, repetition opens up the possibility of being rewarded or punished in the future for current behavior, and game theorists have provided a number of theories to explain the obvious intuition that if the game is repeated often enough, the suspects ought to cooperate.

A practical example of this game could be an Arms Race. Two countries engage in an expensive arms race. They both would like to spend their money on other things (say education). But if one spends the money on education and the other country engages in arms build-up, the weak country will get invaded. This game was played a lot during the cold war. The missile defense of the US is interpreted by some observers as a Prisoner's Dilemma. Player1 (US) can either not build a missile defense system (corresponds to strategy Don't Confess) or build one (strategy Confess). Player2 (Russia) can either not build any more missiles (strategy D) or build more (strategy C). If the US does not build a missile defense system, and Russia does not build more missiles (D,D) then both countries are fairly well off. If Russia builds more missiles and the US has no defense then the US feels very unsafe (C,D). If the US builds a missile shield, and Russia does not produce more missiles then US is happy, but Russia feels unsafe (D,C). If both US and Russia increase their defense budget (C,C), they are equally unsafe, but they are much less well off.