Applied Math 226

Computational Assignment 1.

In this computational assignment we will consider a Markov chain approximation for a two dimensional controlled diffusion. The dynamics are defined by the controlled SDE

$$\begin{aligned} dX_1(t) &= X_2(t)dt + u(t)dt + \sigma_1 dw_1(t) \\ dX_2(t) &= -X_1(t)dt + \sigma_2 dw_2(t), \end{aligned}$$

where the control takes values in $[-1,1]$. The state space is the square $G = (-1,1)^2$. The cost to be minimized is

$$E_x \left[\int_0^\tau \left[c(X(t)) + \frac{b}{2} u(t)^2 \right] dt + g(X(\tau)) \right],$$

where c and g are continuous (do we need c to be positive here?), and τ is the time of first exit from G. The data σ_1, σ_2, b, c and g are supplied by the user, but we can assume bounds for each of these. Suppose we want to approximate by a Markov chain on the grid $hZ^2 \cap G$, where $1/h$ is an integer and also supplied by the user.

At this time we will assume the lower bound $\sigma_1 \geq 1$ (later on we will consider how to get rid of this condition). Then the following Markov chain provides a locally consistent approximation. First let $Q^h(x) = \sigma_1^2 + \sigma_2^2 + h|x_1| + h|x_2|$, and then define the time interpolation by $\Delta t^h(x) = h^2/Q^h(x)$. The controlled transition probabilities are given by

$$p^h(x, x \pm e_1 h|\alpha) = \frac{\frac{1}{2}\sigma_1^2 + h x_2^{\pm} \pm h \frac{1}{2}\alpha}{Q^h(x)}$$

$$p^h(x, x \pm e_2 h|\alpha) = \frac{\frac{1}{2}\sigma_2^2 + h(-x_1)^{\pm}}{Q^h(x)}$$

where $a^{\pm}$ is a if $a \geq 0$ and $-a$ if $a \leq 0$. (You should check that these are probabilities, and that "local consistency" holds.) The DPE is then

$$V^h(x) = \inf_{\alpha \in [-1,1]} \left[\left[c(x) + \frac{b}{2}\alpha^2 \right] \Delta t^h(x) + \sum_{y \in hZ^2 \cap G} p^h(x, y|\alpha) V^h(y) \right],$$

together with the boundary conditions $V^h(x) = g(x)$ for $x \in hZ^2 \cap \partial G$.

A simple iterative solver for this problem is

$$V_{n+1}^h(x) = \inf_{\alpha\in[-1,1]} \left[\left[c(x) + \frac{b}{2}\alpha^2 \right] \Delta t^h(x) + \sum_{y\in hZ^2\cap G} p^h(x,y|\alpha) V_n^h(y) \right] \wedge M$$

and $V_{n+1}^h(x) = g(x)$ for $x \in hZ^2 \cap \partial G$, where M is some large constant. (One is also interested in the related Gauss-Seidel iteration). Write a Matlab code to implement this iterative scheme.