Applied Math 226

Assignment 3.

In this problem I would like you to construct two different types of Markov chain approximations for a finite time problem by reducing it to an exit time problem with one dimension more. The setup is the following. We consider a controlled diffusion (one dimension for simplicity) of the form

$$dX = b(X,u)dt + \sigma(X)dw,$$

with the usual assumptions on the control space, coefficients, etc. We consider the following cost:

$$V(x) = \inf E_x \left[\int_0^{\tau \wedge 1} k(X(s),u(s))ds + g(X(\tau \wedge 1)) \right],$$

where $\tau = \inf \{s : |X(s)| = 1\}$.

This is a control problem on a finite time horizon, with the exit cost $g(x), x = \pm 1$, and the terminal cost $g(x), -1 < x < 1$ if exit has not occured by time 1. For simplicity we will assume both k and g are continuous, though this is not needed (and often g will not be continuous). It is useful to consider the problem

$$V(x,t) = \inf E_{x,t} \left[\int_t^{\tau \wedge 1} k(X(s),u(s))ds + g(X(\tau \wedge 1)) \right],$$

which formally satisfies the PDE

$$V_t + \inf_{\alpha \in U} [L^\alpha V + k] = 0$$

the boundary conditions

$$V(1,t) = g(1), V(-1,t) = g(-1), 0 \le t \le 1$$

and the terminal condition

$$V(x,1) = g(x), -1 < x < 1.$$

Interpret this as a two dimensional escape time problem, by appending time as another state variable. In particular, time will be the second state variable. Then consider the standard grid with spacings h_1 and h_2 in the two dimensions, and construct two Markov chain approximations with the following properties. For the first chain a state of the form (x, t) is allowed to jump only to states of the form $(x+h_1, t+h_2), (x-h_1, t+h_2)$, and $(x, t+h_2)$. For the second chain a state of the form (x, t) is allowed to jump only to states of the form $(x+h_1, t), (x-h_1, t)$, and $(x, t+h_2)$. How are the grid spacings related in the two cases? The first form is called an explicit scheme, while the second is called an implicit scheme. Write down the corresponding DPEs. Is there an obvious order when Gauss-Seidel is used?