Applied Math 226

Assignment 2.

1. In class, the following definition of an admissible control process was given. A process $\{(X_i, u_i), i = 0, 1, ...\}$ taking values in $S \times U$ was called admissible if there was a controlled transition function $p(x, y|u)$ such that for any $n = 1, 2, ...,$

$$\begin{aligned} P(X_{n+1} = y|(X_i, u_i), i = 0, 1, ..., n) &= P(X_{n+1} = y|X_n, u_n) \\ &= p(X_n, y|u_n). \end{aligned}$$

   In class we also remarked that pure Markov controls generated an admissible process, in that if $F_n : S \to U$ are given deterministic functions, if the chain X_n is generated by the (possibly non-stationary) transition function $p(x, y|F_n(x))$, and if $u_n = F_n(X_n)$, then

$$\{(X_i, u_i), i = 0, 1, ...\}$$

   is admissible. (Note that we often abuse notation and write $u_n(x)$ (or $u(x)$ when it does not depend on n) for $F_n(x)$, but $u_n(x)$ should not be confused with u_n.)

   Suppose we want to allow for randomized controls. In this setting, we are given functions $G_n : S \to P(U)$, where $P(U)$ are the probability distributions on U. Given an initial condition $X_0 = x$ (this can also be replaced by an initial distribution), we construct (X_n, u_n) recursively as random variables with (conditional) distributions

$$P(u_n \in A|X_n) = G_n(X_n, A)$$

$$P(X_{n+1} = y|X_n, u_n) = p(X_n, y|u_n)$$

   for Borel sets A. Is (X_n, u_n) is admissible?

2. In class we considered the problem of control till exit, with the cost

$$V(x) = \inf E_x \left[\sum_{i=0}^{N-1} c(X_i, u_i) + g(X_N) \right].$$

Here $N = \min\{i : X_i \in \partial S\}$, $p(x, y|u)$ is continuous in $u \in U$, $c(x, u)$ is positive and continuous in u, and we assumed there was at least one feedback control $u_0 : S - \partial S \to U$ that made $R(u_0)(x, y) = p(x, y|u_0(x))$ a contraction. We also proved the existence of a solution to the associated dynamic programming equation

$$
\begin{aligned}
\bar{V}(x) &= \min_{u \in U} \left[c(x, u) + \sum_{y \in S} p(x, y|u) \bar{V}(y) \right], x \in S - \partial S \\
\bar{V}(x) &= g(x), x \in \partial S.
\end{aligned}
$$

Complete the proof that the minimal cost is equal to the solution to the DPE, and that the DPE has a unique solution.