Applied Math 226

Assignment 1.

1. Consider the optimal stopping problem where one must stop by a fixed deterministic time M. In other words, we have real valued costs c and g defined on S (note that we will no longer need c positive) and consider the cost

$$V(x, M) = \inf E_x \left[\sum_{i=0}^{(M\wedge N)-1} c(X_i) + g(X_{M\wedge N}) \right].$$

Here the infimum is over a class of admissible controls, which we will take to be the pure Markov controls as defined in class. Consider also the DPE

$$\begin{aligned} \bar{V}(x, M) &= \min \left[g(x), c(x) + \sum_{y\in S} p(x, y)\bar{V}(y, M-1) \right], \\ \bar{V}(x, 0) &= g(x). \end{aligned}$$

Does this DPE have a well defined solution? Prove (via a verification theorem) that $\bar{V}(x, M) = V(x, M)$ for all $x \in S$ and all nonnegative integers M.

2. Let $R = r(x, y)$ be a substochastic matrix, which means that $r(x, y) \geq 0$ and that $\sum_{y\in S} r(x, y) \leq 1$ for all $x \in S$. Thus each row sum of R is bounded by 1. Let us assume an additional property, namely that for some finite integer k, each row sum of R^k is strictly less than 1. Consider the vector-matrix equation

$$W_{n+1} = RW_n + C,$$

where an initial condition $W_0 = f$ is given, and C is a column vector of costs. Exhibit W_n as the cost associated with a Markov chain. Does W_n converge? What is the limit? (Hint: You may want to append a state to S.)

3. Consider the optimal stopping problem discussed in class:

$$V(x) = \inf E_x \left[\sum_{i=0}^{N-1} c(X_i) + g(X_N) \right].$$

(Here we assume that c is positive.) In class we proved optimality among all pure Markov controls. Now consider stopping times N such that for any nonnegative integers $m > n$ and bounded and Borel function F,

$$E\left[F(X_{n+1},...,X_m)|X_i, i \leq n, N1_{\{N\leq n\}}\right] = E\left[F(X_{n+1},...,X_m)|X_n\right].$$

We first consider a randomized feedback control, constructed as follows. Let $\{\{U_i(x), x \in S\}, i = 0, 1, ...\}$ be a iid random vector fields that take values in $\{0, 1\}$ and which are independent of the chain $\{X_i\}$. Suppose that N is defined by $N = \min\{i : U_i(X_i) = 1\}$. Show that N is admissible in the sense defined above. Then prove (verification theorem) that V coincides with the unique solution to the DPE

$$\bar{V}(x) = \min\left[g(x), c(x) + \sum_{y\in S} p(x,y)\bar{V}(y)\right].$$