Applied Mathematics 120, Spring of 2005

Derivation of the Dynamic Programming Equation for an Example Done in Class

A detailed derivation of the dynamic programming equation. The key is that we can think of the final position x_N in two ways. One is as a function of $(u_n, \ldots, u_{N-1})$ and x_n, and the second is as a function of $(u_{n+1}, \ldots, u_{N-1})$ and x_{n+1}, which itself depends on u_n and x_n :

$$
\begin{aligned}
&V_n(x_n)\\
&= \max_{(u_n,\ldots,u_{N-1})} \left[\sum_{j=n}^{N-1} B\left[1-u_j\right] + g\left(x_N\right) \right]\\
&= \max_{(u_n,\ldots,u_{N-1})} \left[B\left[1-u_n\right] + \sum_{j=n+1}^{N-1} B\left[1-u_j\right] + g\left(x_N\left[x_n, u_n, \ldots, u_{N-1}\right]\right) \right]\\
&= \max_{(u_n,\ldots,u_{N-1})} \left[B\left[1-u_n\right] + \sum_{j=n+1}^{N-1} B\left[1-u_j\right] + g\left(x_N\left[x_{n+1}\left[x_n, u_n\right], u_{n+1}, \ldots, u_{N-1}\right]\right) \right]\\
&= \max_{u_n} \left[B\left[1-u_n\right] + \max_{(u_{n+1},\ldots,u_{N-1})} \left[\sum_{j=n+1}^{N-1} B\left[1-u_j\right] + g\left(x_N\left[x_{n+1}\left[x_n, u_n\right], u_{n+1}, \ldots, u_{N-1}\right]\right) \right] \right]\\
&= \max_{u_n} \left[B\left[1-u_n\right] + \max_{(u_{n+1},\ldots,u_{N-1})} \left[\sum_{j=n+1}^{N-1} B\left[1-u_j\right] + g\left(x_N\left[f\left(x_n, u_n\right), u_{n+1}, \ldots, u_{N-1}\right]\right) \right] \right]\\
&= \max_{u_n} \left[B\left[1-u_n\right] + V_{n+1}\left(f\left(x_n, u_n\right)\right) \right].
\end{aligned}
$$

The first equality is the definition of $V_n(x_n)$, the second separates out the benefit for the first step after n and writes x_N as a function of $(u_n, \ldots, u_{N-1})$ and x_n, the third writes x_N as a function of $(u_{n+1}, \ldots, u_{N-1})$ and $x_{n+1}\left[x_n, u_n\right]$, the fourth uses that the one step benefit does not depend on $u_{n+1}, \ldots, u_{N-1}$, the fifth uses the formula for how we get x_{n+1} from x_n, u_n, and the last uses the definition of V_{n+1}.