Method of undetermined coefficients

Superposition Rule. If the non-homogeneous term $f(x)$ in

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = f(x), \tag{1}$$

can be broken down into the sum $f(x) = f_1(x) + f_2(x) + \cdots + f_m(x)$, where each term $f_j(x),\ j = 1, 2, \ldots, m$ is a function of the form

$$f(x) = P_k(x)\, e^{\alpha x} \cos \beta x + Q_k(x)\, e^{\alpha x} \sin \beta x, \tag{2}$$

apply the rules described below to obtain a particular solution $y_{pj}(x)$ of the non-homogeneous equation with each non-homogeneous term $f_j(x)$. Then the sum of these particular solutions $y_p(x) = y_{p1}(x) + \cdots + y_{pm}(x)$ is a particular solution of equation for $f(x) = f_1(x) + f_2(x) + \cdots + f_m(x)$.

Basic Rule: *When the control number is not a root of the characteristic equation for the associated homogeneous equation.* If $f(x)$ in Eq.(1) is the function of the form (2) and its control number is **not** a root of the characteristic equation $L[\lambda] = 0$, choose the corresponding particular solution in the **same form** as $f(x)$. In particular, if $f(x)$ is one of the functions in the first column in Table times a constant, and it is not a solution of the homogeneous equation $L[y] = 0$, choose the corresponding function y_p in the third column as a particular solution of Eq.(1) and determine its coefficients (which is denoted by C) by substituting y_p and its derivatives into Eq.(1).

Modification Rule: *When the control number* **is** *a root of the characteristic equation.* If the control number of the right-hand side function in Eq.(1) is a root of the characteristic equation of multiplicity r, choose the corresponding particular solution in the same form multiplied by x^r. In other words, if a term in your choice for y_p happens to be a solution of the corresponding homogeneous equation $L[y] = 0$, then multiply your choice of y_p by x to that power which indicates the multiplicity of the corresponding root of the characteristic equation.

Term in $f(x)$	σ	Choice for $y_p(x)$
$e^{\gamma x}$	γ	$Ce^{\gamma x}$
$x^n\ (n=0,1,\ldots)$	0	$C_0x^n + C_1x^{n-1} + \cdots + C_{n-1}x + C_n$
$x^n\, e^{\gamma x}$	γ	$(C_0x^n + C_1x^{n-1} + \cdots + C_{n-1}x + C_n)e^{\gamma x}$
$\cos ax$	ia	$C_1 \cos ax + C_2 \sin ax$
$\sin ax$	ia	$C_1 \cos ax + C_2 \sin ax$
$x^n\ \cos ax$	ia	$(C_0x^n + C_1x^{n-1} + \cdots + C_{n-1}x + C_n) \cos ax +$
		$+(C_0^1x^n + C_1^1x^{n-1} + \cdots + C_{n-1}^1x + C_n^1) \sin ax$
$x^n\ \sin ax$	ia	$(C_0x^n + C_1x^{n-1} + \cdots + C_{n-1}x + C_n^1) \cos ax +$
		$+(C_0^1x^n + C_1^1x^{n-1} + \cdots + C_{n-1}^1x + C_n^1) \sin ax$
$e^{\alpha x} \cos \beta x$	$\alpha + i\beta$	$e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$
$e^{\alpha x} \sin \beta x$	$\alpha + i\beta$	$e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$
$x^n\, e^{\alpha x} \cos \beta x$	$\alpha + i\beta$	$e^{\alpha x}\ \cos \beta x(C_0x^n + C_1x^{n-1} + \cdots + C_{n-1}x + C_n) +$
		$+e^{\alpha x}\ \sin \beta x(C_0^1x^n + C_1^1x^{n-1} + \cdots + C_{n-1}^1x + C_n^1)$
$x^n\, e^{\alpha x} \sin \beta x$	$\alpha + i\beta$	$e^{\alpha x}\ \cos \beta x(C_0x^n + C_1x^{n-1} + \cdots + C_{n-1}x + C_n) +$
		$+e^{\alpha x}\ \sin \beta x(C_0^1x^n + C_1^1x^{n-1} + \cdots + C_{n-1}^1x + C_n^1)$

Table 1: Method of undetermined coefficients. σ is the control number of the function in the first column.