Week 1

Vladimir Dobrushkin

A **differential equation** is an equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives or differentials of various orders. In this course, we consider only functions depending on only one variable. Correspondingly, we need a special definition. An **ordinary differential equation** (ODE) is a differential equation in which the unknown function (also known as the dependent variable) is a function of a single independent variable. In the simplest form, the unknown function is a real or complex valued function.

Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and/or time (expressed as derivatives) is known or postulated.

Now there is a couple of words about notation. In this course, we will usually denote by x or t an independent variable, and by y a dependent (unknown) variable. However, sometimes other letters will be used. For derivatives, we use either Leibniz' notation (1676): dy/dx, d^2y/dx^2, $\ldots$, or the prime notation: y', y'', $\ldots$. For higher derivatives, we use the notation $y^{(n)}$ to denote the derivative of the order n. When a function depends on time, it is common to denote its first two derivatives with respect to time with dots: $\dot{y}$ and $\ddot{y}$. Recall that the differential dx means an infinitely small change in x. The differential of the function $y = \phi(x)$ is written as $dy = \phi'(x)\,dx$ provided that the function $\phi(x)$ is differentiable. The expression $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$ is treated as a fraction rather than as an operator. Here Δy is the change in value of the function y caused by a small change in x, written as Δx.

Example 1.1. The differential equation

$$\frac{dy}{dx} = 3x^2 - 2x + 1$$

can be written using differentials:

$$dy = \left(3x^2 - 2x + 1\right) dx$$

Since the right-hand side function $(3x^2 - 2x + 1)$ does not depend on y, we can integrate it to obtain

$$y = x^3 - x^2 + x + C,$$

where C is a constant of integration. Note that when we integrate

$$\int dy = \int \left(3x^2 - 2x + 1\right) dx$$

we should get two arbitrary constants on both sides (K and C):

$$y + K = x^3 - x^2 + x + C.$$

But we can combine them into one constant, say $C' = C - K$, which we write on the right side again as C (the difference of two arbitrary constants is again an arbitrary constant). $\square$

The **order** of a differential equation is the order of the highest derivative that appears in the equation. In particular, an ordinary differential equation of the n-th order is an equation of the following form:

$$F\left(x, y(x), y'(x), \ldots, y^{(n)}(x)\right) = 0.$$

here $y(x)$ is a smooth function to be determined, and $F(x, y, p_1, p_2, \ldots, p_n)$ is a given function of $n+2$ variables. If this equation can be solved for $y^{(n)}(x)$, then we obtain the differential equation in the **normal form:**

$$y^{(n)}(x) = f\left(x, y, y', \ldots, y^{(n-1)}\right), \qquad a < x < b.$$

A first order differential equation is of the form

$$F(x, y, y') = 0.$$

In this part of the course, we will consider first order equations in normal form:

$$\frac{dy}{dx} = f(x, y) \qquad \text{or} \qquad dy = f(x, y)\, dx. \tag{1.1}$$

A **solution** or **integral** of the ordinary differential equation

$$F\left(x, y(x), y'(x), \ldots, y^{(n)}(x)\right) = 0$$

on an interval $|a, b|$ $(a < b)$ is a continuous function $y(x)$ such that y, y', y'', $\ldots$, $y^{(n)}$ exist and satisfy the equation for all values of the independent variable on the interval, $x \in |a, b|$. The graphs of the solutions of a differential equation are called their **integral curves** or **streamlines**. $\triangleleft$

A solution in which the dependent variable is expressed in terms of the independent variable is said to be in **explicit form**. A function is *known* if it can be expressed by a formula in terms of standard and/or familiar functions. However, we shall see in this course that functions studied in calculus are not enough to describe solutions of all differential equations. Any relation, free of derivatives, that involves two variables x and y and that is consistent with the differential equation (1.1) is said to be a solution of the equation in **implicit form**.

In this course, you will learn how to determine solutions explicitly or implicitly, how to approximate them numerically, how to visualize and plot solutions, and much more.

Next, we observe that a differential equation may (and usually will) have an infinite number of solutions. A set of solutions of $y' = f(x, y)$ that depends on one arbitrary constant C deserves a special name.

A function $y = \phi(x, C)$ is called the **general solution** to the differential equation $y' = f(x, y)$ in some two-dimensional domain Ω if for every point $(x, y) \in \Omega$ there exists a value of constant C such that the function $y = \phi(x, C)$ satisfies the equation $y' = f(x, y)$. A solution of this differential equation can be defined implicitly:

$$\Phi(x, y, C) = 0 \qquad \text{or} \qquad \psi(x, y) = C. \tag{1.2}$$

In this case, $\Phi(x, y, C)$ is called the **general integral**, and $\psi(x, y)$ is referred to as the **potential function** of the given equation $y' = f(x, y)$. $\triangleleft$

A constant C may be given any value on a suitable range. Since C can vary from problem to problem, it is often called a parameter to distinguish it from the main variables x and y. Therefore, the equation $\Phi(x, y, C) = 0$ defines a one parameter family of curves with no intersections. Graphically, it represents a family of solution curves in the xy-plane, each element of which is associated with a particular value of C.

Sometimes, instead of the general solution that includes an entire family of integral curves corresponding to the differential equation $y' = f(x, y)$, we are interested in one specific curve that goes through a particular point in the plane. A solution to a differential equation is called a **particular** (or specific) **solution** if it does not contain any arbitrary constant. There two general ways to specify a solution out of the general solution: either set a particular value for the constant C or identify a point on the plane and seek a solution that goes through this point.

A differential equation $y' = f(x, y)$ *(or, in general,* $F(x, y, y') = 0$*)* subject to the initial condition $y(x_0) = y_0$, where x_0 and y_0 are specified values, is called an **initial value problem** *(IVP)* or a Cauchy problem. $\triangleleft$

A **singular solution** of $y' = f(x, y)$ is a function that is not a special case of the general solution and for which the uniqueness of the Cauchy problem has failed. $\triangleleft$

The set of all solutions (singular solutions together with the general solution) is sometimes called the *complete solution*. Not every differential equation has a singular solution, but if it does, its singular solution cannot be determined from the general solution by setting a particular value of C, including $\pm\infty$. A differential equation may have a solution that is neither singular nor a member of the family of one-parameter curves from the general solution.

A **nullcline** for the differential equation $y' = f(x, y)$ is the curve where slope is zero: $f(x, y) = 0$. When the nullcline is a constant, then it is always a solution of the given differential equation. A constant solution $y \equiv y^*$ of the differential equation $y' = f(x, y)$ is called the **equilibrium solution**, that is, if $f(x, y^*) \equiv 0$ for all x. In general, a nullcline is not a solution. From calculus it is known that the point where the derivative is zero is called the critical point. Therefore the nullcline is the set of points where solutions may attain extreme values.

When every solution that starts "near" a critical point moves away from it, we call this constant solution an **unstable equilibrium solution** or unstable critical point. We refer to it as a *repeller* or *source*. If every solution that starts "near" a critical point moves toward the equilibrium solution, we call it an **asymptotically stable equilibrium solution**. We refer to it as an *attractor* or *sink*. When solutions on one side of the equilibrium solution move towards it and on the other side of the constant solution move away from it, we call the equilibrium solution semi-stable.

A **slope field** or **direction field** is a graphical representation of the solutions of a first-order differential equation. It is achieved without solving the differential equation analytically, and thus it is useful. The representation may be used to qualitatively visualize solutions, or to numerically approximate them.

At any point (x, y), the slope dy/dx of the solution $y(x)$ at that point is given by $f(x, y)$. We can indicate this by drawing a short line segment (or arrow) through the point (x, y) with the slope $f(x, y)$. The collection of all such line segments at each point (x, y) of a rectangular grid of points is called a **direction field** or a **slope field** of the differential equation $y' = f(x, y)$. By increasing the density of arrows, it would be possible, in theory at least, to approach a limiting curve—a streamline (also called the trajectory).

When high precision is required, a suitably dense set of line segments on the plane region must be made. The labor involved may then be substantial. Fortunately, available software packages are very helpful for practical drawings of direction fields instead of hand sketching. Therefore, we present some of the most popular packages.

There is a friendly graphical program, **Winplot**, written by Richard Parris: `http://math.exeter.edu/rparris/winplot.html`.

Maple has two dedicated commands for plotting flow fields associated with a first order differential equation—`DEplot` and `dfieldplot`. For example, the commands

```
restart;  with(DEtools):  with(plots):
dfieldplot(diff(y(x),x)=y(x)*y(x) -x*x, y(x), x=-1..1, y=-2..2, arrows=medium);
```

allow you to plot the direction field for the differential equation $y' = y^2 - x^2$. To include graphs of some solutions into the direction field, we define the initial conditions first:

```
inc:=[y(0)=0.5,y(0)=-1];
```

Then we type

```
DEplot(diff(y(x),x)=y(x)*y(x) -x*x, y(x), x=-1..1, y=-2..2, inc, arrows=medium,
linecolor=black,color=blue,title=‘Direction field for y’=y+x‘);
```

With Mathematica-7 or a later version, only one command is needed to draw the direction field corresponding to the equation $y' = y^2 - t^2$:

```
dfield = VectorPlot[{1,y^2-t^2}, {t, -2, 2}, {y, -2, 2}, Axes -> True,
VectorScale -> {Small,Automatic,None}, AxesLabel -> {"t", "dydt=y^2-t^2"}]
```

MATLAB does not have the ability to plot direction fields naturally; so we present Examples of codes in GNU Octave/MATLAB

```
Ffun = @(X,Y)X.*Y;                % function f(x,y)=xy
[X,Y]=meshgrid(-2:.3:2,-2:.3:2); % choose the plot sizes
DY=Ffun(X,Y); DX=ones(size(DY)); % generate the plot values
quiver(X,Y,DX,DY);                % plot the direction field
hold on;
contour(X,Y,DY,[-6 -2 -1 0 1 2 6]); %add the isoclines
title('Slope field and isoclines for f(x,y)=xy')
```

Alternate example code in GNU Octave/MATLAB

```
funn = @(x,y)y-x;                 % function f(x,y)=y-x
[x,y]=meshgrid(-2:0.5:2);         % intervals for x and y
slopes=funn(x,y);                 % matrix of slopes
dy=slopes./sqrt(1+slopes.^2);     % normalize the line element...
dx=sqrt(1-dy.^2);                 % ...magnitudes for dy and dx
quiver(x,y,dx,dy);                % plot the direction field
```

Many universities have developed some software packages that facilitate drawing direction fields for differential equations. For example, John Polking at Rice University has produced `dfield` and `pplane` programs for MATLAB. The MATLAB versions of `dfield` and `pplane` are copyrighted in the name of John Polking. They are not in the public domain. However, they are being made available free for use in educational institutions. Another option may be to use a friendly package Iode (`http://www.math.uiuc.edu/iode/`) that runs under either MATLAB or Octave packages.