APMA 0330 — Applied Mathematics - I

Brown University **Fall, 2017**

Solutions to Homework, Set 8 **Due December 6**

8.1 (40 pts) Find the inverse Laplace transforms of the following functions.

(a) (10 pts) $\mathcal{L}^{-1}\left[\frac{\lambda^2-9}{\left(\lambda^2+9\right)^2}\,e^{-\lambda}\right] = (t-1)\,\cos(3t-3)\,H(t-3);$

(b) (10 pts) $\mathcal{L}^{-1}\left[\frac{2}{(\lambda-2)^2+4}\right] = e^{2t}\,\sin 2t\,H(t);$

(c) (10 pts) $\mathcal{L}^{-1}\left[\frac{5}{\lambda^2+\lambda-6}\right] = \left[e^{2t}-e^{-3t}\right]H(t);$

(d) (10 pts) $\mathcal{L}^{-1}\left[\frac{32}{(\lambda-3)^2\,(\lambda+1)^2}\right] = \left[e^{-t}\,(1+2t)+e^{3t}\,(2t-1)\right]H(t).$

Solution: In all problems, we first determine singular points that are nulls of the denominator.

(a) The denominator λ^2+9 has two complex conjugate roots $\lambda = \pm 3\mathbf{j}$ of multiplicity 2. To find the inverse Laplace transform, we calculate only one residue:

$$\begin{aligned}
p(t) &= \operatorname*{Res}_{\lambda=3\mathbf{j}} \frac{\lambda^2-9}{\left(\lambda^2+9\right)^2}\,e^{\lambda t} = \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\frac{\lambda^2-9}{\left(\lambda+3\mathbf{j}\right)^2}\,e^{\lambda t}\right|_{\lambda=3\mathbf{j}} \\
&= \left[\frac{2\lambda}{\left(\lambda+3\mathbf{j}\right)^2} - 2\,\frac{\lambda^2-9}{\left(\lambda+3\mathbf{j}\right)^3} + t\,\frac{\lambda^2-9}{\left(\lambda+3\mathbf{j}\right)^2}\right]_{\lambda=3\mathbf{j}} e^{3\mathbf{j}t} \\
&= \left[\frac{6\mathbf{j}}{\left(6\mathbf{j}\right)^2} - 2\,\frac{-9-9}{\left(6\mathbf{j}\right)^3} + t\,\frac{-9-9}{\left(6\mathbf{j}\right)^2}\right] e^{3\mathbf{j}t} \\
&= \left[\frac{1}{6\mathbf{j}} - \frac{1}{6\mathbf{j}} + \frac{t}{2}\right] e^{3\mathbf{j}t} = \frac{t}{2}\,e^{3\mathbf{j}t}.
\end{aligned}$$

Extracting the real part and multiplying by two, we obtain

$$\mathcal{L}^{-1}\left[\frac{\lambda^2-9}{\left(\lambda^2+9\right)^2}\right] = t\,\cos 3t\,H(t).$$

Since multiplication by an exponential multiple corresponds to the shift, we get the required formula.

Other parts of this exercise follow the same pattern.

8.2 (20 pts) Using the Laplace transform, solve the initial value problem.

$$2\,y'' - 7\,y' + 3\,y = H(t) - H(t-2), \qquad y(0) = 0, \quad y'(0) = 1.$$

Solution: Application of the Laplace transform to the given initial value problems gives

$$\left(2\lambda^2 - 7\lambda + 3\right) y^L(\lambda) - 2\left(y'(0) + \lambda\, y(0)\right) + 7\, y(0) = \frac{1}{\lambda}\left[1 - e^{-2\lambda}\right],$$

where y^L is the Laplace transform of the unknown function. Upon some simplification, we get

$$y^L(\lambda) = \frac{2}{2\lambda^2 - 7\lambda + 3} + \frac{1}{2\lambda^2 - 7\lambda + 3}\,\frac{1}{\lambda}\left[1 - e^{-2\lambda}\right].$$

The first term, which we denote by y_h^L, is

$$y_h^L(\lambda) = \frac{2}{2\lambda^2 - 7\lambda + 3} = \frac{2}{(2\lambda -)(\lambda - 3)} = \frac{1}{5(\lambda - 3)} - \frac{2}{5(2\lambda - 1)}.$$

Its inverse Laplace transform

$$y_h(t) = \mathcal{L}^{-1}\left[\frac{2}{2\lambda^2 - 7\lambda + 3}\right] = \frac{2}{5}\left(e^{3t} - e^{t/2}\right) H(t)$$

is the solution of the initial value problem

$$2\,y'' - 7\,y' + 3\,y = 0, \qquad y(0) = 0, \quad y'(0) = 1.$$

The second term, which we denote by y_p^L, is the difference of two functions:

$$y_p^L = \frac{1}{2\lambda^2 - 7\lambda + 3}\,\frac{1}{\lambda}\left[1 - e^{-2\lambda}\right].$$

Taking the inverse Laplace transform, we represent this function as

$$y_p(t) = \mathcal{L}^{-1}\left[\frac{1}{2\lambda^2 - 7\lambda + 3}\,\frac{1}{\lambda}\left(1 - e^{-2\lambda}\right)\right] = g(t) - g(t-2),$$

where

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{2\lambda^2 - 7\lambda + 3}\,\frac{1}{\lambda}\right].$$

Application of the residue theorem yields

$$g(t) = \operatorname*{Res}_{\lambda=0} + \operatorname*{Res}_{\lambda=1/2} + \operatorname*{Res}_{\lambda=3} \frac{e^{\lambda t}}{\lambda(2\lambda - 1)(\lambda - 3)}.$$

Each residue is not hard to evaluate:

$$\operatorname*{Res}_{\lambda=0} = \lim_{\lambda\mapsto 0} \frac{e^{\lambda t}}{2\lambda^2 - 7\lambda + 3} = \frac{1}{3},$$
$$\operatorname*{Res}_{\lambda=1/2} = \lim_{\lambda\mapsto 1/2} \frac{e^{\lambda t}}{2\lambda(\lambda - 3)} = -\frac{2}{5}\, e^{t/2},$$
$$\operatorname*{Res}_{\lambda=3} = \lim_{\lambda\mapsto 3} \frac{e^{\lambda t}}{\lambda(2\lambda - 1)} = \frac{1}{15}\, e^{3t}.$$

Adding these three functions, we obtain

$$g(t) = \left[\frac{1}{3} - \frac{2}{5}\, e^{t/2} + \frac{1}{15}\, e^{3t}\right] H(t).$$

Note that the function $g(t)$ is the solution of the initial value problem (with homogeneous initial conditions):

$$2\, y'' - 7\, y' + 3\, y = H(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

8.3 (40 pts) Using the Laplace transform, solve the initial value problem.

$$y'' - 2\, y' + 5\, y = \sin(2t) \left[H(t - \pi) - H(t - 5\pi)\right], \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution: Applying the Laplace transform, we reduce the given initial value problem to the algebraic equation for y^L, the Laplace transform of the unknown solution:

$$\left(\lambda^2 - 2\,\lambda + 5\right) y^L = \frac{2}{\lambda^2 + 4}\left[e^{-\pi\lambda} - e^{-5\pi\lambda}\right].$$

We can express $y(t)$ via one function

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{\lambda^2 - 2\,\lambda + 5}\, \frac{2}{\lambda^2 + 4}\right]$$

as

$$y(t) = g(t - \pi) - g(t - 5\pi).$$

Since the denominator has two pairs of complex conjugate roots

$$\lambda = 1 \pm 2\mathbf{j} \qquad \text{and} \qquad \lambda = \pm 2\mathbf{j},$$

we just need to find two residures:

$$\operatorname*{Res}_{1+2\mathbf{j}} \frac{e^{\lambda t}}{\lambda^2 - 2\lambda + 5} \cdot \frac{2}{\lambda^2 + 4} = \lim_{\lambda\mapsto 1+2\mathbf{j}} \frac{e^{\lambda t}}{2\lambda - 2} \cdot \frac{2}{\lambda^2 + 4}$$
$$= \frac{1}{2}\, \frac{e^{\lambda t}}{\lambda - 1} \cdot \frac{2}{\lambda^2 + 4}\bigg|_{\lambda=1+2\mathbf{j}} = \frac{1}{2}\, e^t\, \frac{e^{2\mathbf{j}t}}{2\mathbf{j}} \cdot \frac{2}{1 + 4\mathbf{j}}$$
$$= \frac{e^t}{2}\, \frac{e^{2\mathbf{j}t}}{\mathbf{j}} \cdot \frac{1 - 4\mathbf{j}}{1 + 4^2} = \frac{e^t}{2 \cdot 17}\, e^{2\mathbf{j}t} \left(\frac{1}{\mathbf{j}} - 4\right),$$

and similarly

$$\operatorname*{Res}_{2\mathbf{j}} \frac{e^{\lambda t}}{\lambda^2 - 2\lambda + 5} \cdot \frac{2}{\lambda^2 + 4} = \lim_{\lambda \mapsto 2\mathbf{j}} \frac{e^{\lambda t}}{\lambda^2 - 2\lambda + 5} \cdot \frac{2}{2\lambda}$$
$$= \frac{e^{2\mathbf{j}t}}{2\mathbf{j}} \cdot \frac{1}{1 - 4\mathbf{j}} = \frac{e^{2\mathbf{j}t}}{2\mathbf{j}} \cdot \frac{1 + 4\mathbf{j}}{17}.$$

Extracting real parts of the above residures, we get

$$\Re \operatorname*{Res}_{1+2\mathbf{j}} \frac{e^{\lambda t}}{\lambda^2 - 2\lambda + 5} \cdot \frac{2}{\lambda^2 + 4} = \frac{e^t}{2 \cdot 17} \left(\sin 2t - 4 \cos 2t\right),$$
$$\Re \operatorname*{Res}_{2\mathbf{j}} \frac{e^{\lambda t}}{\lambda^2 - 2\lambda + 5} \cdot \frac{2}{\lambda^2 + 4} = \frac{1}{2 \cdot 17} \left(\sin 2t + 4 \cos 2t\right).$$

Multiplication by 2 gives

$$g(t) = 2\,\Re \sum \operatorname{Res} \frac{e^{\lambda t}}{\lambda^2 - 2\lambda + 5} \cdot \frac{2}{\lambda^2 + 4} = \frac{1}{17} \left[\left(1 + e^t\right) \sin 2t + 4 \left(1 - e^t\right) \cos 2t\right] H(t).$$