APMA 0330 — Applied Mathematics - I

Brown University **Fall, 2017**

Solutions to Homework, Set 7 **Due November 29**

7.1 (20 pts) Find the general solution to the following differential equation

$$y'' + 9\,y = 3\,\cot\left(3x\right).$$

Solution: The characteristic equation $\lambda^2 + 9 = 0$, corresponding to the homogeneous equation, $y'' + 9\,y = 0$, has two complex conjugate roots $\lambda = \pm 3\mathbf{j}$. Therefore, the general solution of the homogeneous equation is

$$y(x) = C_1 \cos 3x + C_2 \sin 3x,$$

for some arbitrary constants C_1 and C_2. WE seek a particular solution in the form

$$y_p(x) = A(x)\,\cos 3x + B(x)\,\sin 3x$$

for some functions $A(x)$ and $B(x)$. Then their derivatives satisfy the Lagrange system of algebraic equations:

$$\begin{aligned} A'(x)\,\cos 3x + B'(x)\,\sin 3x &= 0, \\ -A'(x)\,\sin 3x + B'(x)\,\cos 3x &= \cot\left(3x\right). \end{aligned}$$

Solving this system of equations, we find

$$A'(x) = -\cos 3x, \qquad B'(x) = \cos 3x\,\cot 3x = \frac{\cos^2 3x}{\sin 3x}.$$

Integration yields

$$A(x) = -\frac{1}{3}\,\sin 3x, \qquad B(x) = \frac{1}{3}\,\cos 3x + \frac{1}{3}\,\ln\left|\tan\frac{3x}{2}\right|.$$

Substituting these values into the formula, we get a particular solution:

$$y_p(x) = \frac{1}{3}\,\cos 3x\,\ln\left|\tan\frac{3x}{2}\right|.$$

Then the general solution of the given equation will be the sum of these two functions:

$$y = y_h(x) + y_p(x) = \frac{1}{3}\,\cos 3x\,\ln\left|\tan\frac{3x}{2}\right| + C_1 \cos 3x + C_2 \sin 3x.$$

7.2 (60 pts) In each of problems, express $f(t)$ in terms of the Heaviside function, $H(t)$, and find its Laplace transform.

(a) $f(t) = \begin{cases} 3, & 0 \le t < 3, \\ -2, & 3 \le t < 5, \\ 1, & 5 \le t < 8, \\ 2, & t \ge 8. \end{cases}$

(b) $f(t) = \begin{cases} 0, & 0 \le t < 1, \\ -3, & 1 \le t < 2, \\ 2, & 2 \le t < 3, \\ -4, & 3 \le t < 4, \\ 1, & t \ge 4. \end{cases}$

(c) $f(t) = \begin{cases} 1, & 0 \le t < 3, \\ e^{2(t-3)}, & t \ge 3. \end{cases}$

(d) $f(t) = \begin{cases} t^2, & 0 \le t < 2, \\ t+1, & t \ge 2. \end{cases}$

(e) $f(t) = \begin{cases} t^2, & 0 \le t < 1, \\ t-1, & 1 \le t < 2, \\ t^2+1, & 2 \le t < 3, \\ 10, & t \ge 3. \end{cases}$

(f) $f(t) = \begin{cases} t-1, & 0 \le t < 2, \\ 1, & 2 \le t < 4, \\ 5-t, & 4 \le t < 8, \\ -3, & t \ge 8. \end{cases}$

Hint: The Laplace transform of the power function is

$$\mathcal{L}\left[t^n\right] = \int_0^\infty t^n e^{-\lambda t}\,\mathrm{d}t = \frac{n!}{\lambda^{n+1}}, \qquad n = 0, 1, 2, \ldots.$$

Solution:

(a) **(5 pts)** First, we represent the given functions as a sum of Heaviside functions:

$$f(t) = 3\left[H(t) - H(t-3)\right] - 2\left[H(t-3) - H(t-5)\right] + H(t-5) - H(t-8) + 2\,H(t-8).$$

Then we simplify

$$f(t) = 3\,H(t) - 5\,H(t-3) + 3\,H(t-5) + H(t-8).$$

According to the shift rule, we get its Laplace transform

$$f^L(\lambda) = \frac{1}{\lambda}\left[3 - 5\,e^{-3\lambda} + 3\,e^{-5\lambda} + e^{-8\lambda}\right].$$

(b) **(5 pts)** First, we represent the given functions as a sum of Heaviside functions:

$$f(t) = -3\left[H(t-1) - H(t-2)\right] + 2\left[H(t-2) - H(t-3)\right] - 4\left[H(t-3) - H(t-4)\right] + H(t-4).$$

Then we simplify

$$f(t) = -3\,H(t-1) + 5\,H(t-2) - 6\,H(t-3) - 3\,H(t-4).$$

According to the shift rule, we get its Laplace transform

$$f^L(\lambda) = \frac{1}{\lambda}\left[-3\,e^{-\lambda} + 5\,e^{-2\lambda} - 6\,e^{-3\lambda} - 3\,e^{-4\lambda}\right].$$

(c) **(5 pts)** We break the given function into the sum of two functions:

$$f(t) = [H(t) - H(t-3)] + e^{2(t-3)}\, H(t-3) = H(t) + \left(e^{2(t-3)} - 1\right) H(t-3).$$

Application of Laplace transform yields

$$f^L(\lambda) = \frac{1}{\lambda}\left[1 - e^{-3\lambda}\right] + \frac{1}{\lambda - 2}\, e^{-3\lambda}.$$

(d) **(5 pts)** We break the given function into the sum of two functions:

$$f(t) = t^2 [H(t) - H(t-2)] + (t+1)\, H(t-2) = t^2 H(t) + \left(t + 1 - t^2\right) H(t-2).$$

We make a shift by 2 in the function

$$t + 1 - t^2 = (t-2) + 3 - (t - 2 + 2)^2 = -(t-2)^2 - 3\,(t-2) - 1.$$

This allows us to represent the function as

$$f(t) = t^2\, H(t) - H(t-2)\left[(t-2)^2 + 3\,(t-2) + 1\right].$$

Then we apply the shift rule:

$$f^L(\lambda) = \frac{2}{\lambda^3} - e^{-2\lambda}\left[\frac{2}{\lambda^3} + \frac{3}{\lambda^2} + \frac{1}{\lambda}\right].$$

(e) **(5 pts)** Using the Heaviside function, we represent the given function as a sum of

$$\begin{aligned} f(t) &= t^2 [H(t) - H(t-1)] + (t-1)\,[H(t-1) - H(t-2)] + (t^2+1)\,[H(t-2) - H(t-3)] + 10\, H(\\ &= t^2 H(t) + \left[t - 1 - t^2\right] H(t-1) + \left[t^2 - t + 2\right] H(t-2) + \left[9 - t^2\right] H(t-3) \\ &= t^2 H(t) + \left[-t + 1 - (t-1)^2 - 1\right] H(t-1) + \left[(t-2)^2 + 3(t-2) + 4\right] H(t-2) - \left[(t-3)^2 + \right. \end{aligned}$$

Application of shift rule yields

$$f^L = \frac{2}{\lambda^3} - \frac{2 + \lambda + \lambda^2}{\lambda^3}\, e^{-\lambda} + \frac{2 + 3\lambda + 4\lambda^2}{\lambda^3}\, e^{-2\lambda} - \frac{2 + 6\lambda}{\lambda^3}\, e^{-3\lambda}.$$

(f) **(5 pts)** Using the Heaviside function, we represent the given function as a sum of

$$\begin{aligned} f(t) &= (t-1)\,[H(t) - H(t-2)] + [H(t-2) - H(t-4)] + (5-t)\,[H(t-4) - H(t-8)] - 3\, H(t - \\ &= (t-1)\, H(t) - (t-2)\, H(t-2) - (t-4)\, H(t-4) + (t-5)\, H(t-8) \\ &= (t-1)\, H(t) - (t-2)\, H(t-2) - (t-4)\, H(t-4) + (t-8)\, H(t-8) \end{aligned}$$

Application of shift rule yields

$$f^L = \frac{1 - \lambda}{\lambda^2} - \frac{1}{\lambda^2}\, e^{-2\lambda} - \frac{1}{\lambda^2}\, e^{-4\lambda} + \frac{1}{\lambda^2}\, e^{-8\lambda}.$$

7.3 (20 pts) Find the Laplace transform of the periodic with period $T = 6$ sawtooth function that is the half-wave rectifier of the function

$$f(t) = 1 - t, \quad 0 < t < 3.$$

Solution: First, we find the integral, which we denote by

$$A = \int_0^3 (1-t)\, e^{-\lambda t} \mathrm{d}t = \frac{1}{\lambda} - \frac{1}{\lambda^2} + \frac{1+2\lambda}{\lambda^2}\, e^{-3\lambda}.$$

Then the Laplace transform of the half-wave rectifier will be

$$\frac{1}{1 - e^{-6\lambda}}\, A.$$