APMA 0330 — Applied Mathematics - I

Brown University **Fall, 2017**

Solutions to Homework, Set 6 **Due November 8**

6.1 (10 pts) Write out the characteristic equation for the given differential equation:

$$(\mathbf{a})\ \ y^{(4)} + 5\,y'' - 3y = 0; \qquad\qquad (\mathbf{b})\ \ y'' - 7\,y' + 2\,y = 0.$$

Solution:

$$(\mathbf{a})\ \ \lambda^4 + 5\,\lambda^2 - 3 = 0; \qquad\qquad (\mathbf{b})\ \ \lambda^2 - 7\,\lambda + 2 = 0.$$

6.2 (10 pts) Let α and β be real constants. Consider the differential operator of the second order:

$$L[\texttt{D}] = (\texttt{D} - \alpha)^2 + \beta^2,$$

where $\texttt{D} = \mathrm{d}/\mathrm{d}t$ is the derivative operator. Show that the substitution $y = e^{\alpha t} v(t)$ reduces the differential equation

$$L[\texttt{D}]y = 0 \qquad \text{or} \qquad \left[(\texttt{D} - \alpha)^2 + \beta^2 \right] y = 0$$

to the canonical equation $\ddot{v} + \beta^2 v = 0$, where dot stands for the derivative with respect to t.

Solution: Calculating derivatives, we get

$$\dot{y} = e^{\alpha t} \left[\alpha\, v + \dot{v} \right], \qquad \ddot{y} = e^{\alpha t} \left[\alpha^2 v + 2\alpha\, \dot{v} + \ddot{v} \right].$$

Then

$$\begin{aligned} L[\texttt{D}]y &= \left[\texttt{D}^2 - 2\alpha\, \texttt{D} + \left(\alpha^2 + \beta^2 \right) \right] e^{\alpha t} v(t) \\ &= e^{\alpha t} \left[\alpha^2 v + 2\alpha\, \dot{v} + \ddot{v} - 2\alpha \left(\alpha\, v + \dot{v} \right) + \left(\alpha^2 + \beta^2 \right) v \right] = e^{\alpha t} \left[\ddot{v} + \beta^2 v \right]. \end{aligned}$$

6.3 (10 pts) The Wronskian of two functions is $W(x) = x^2 - 6x + 9$. Are the functions linearly independent or linearly dependent?

Solution: The Wronskian is $W(x) = (x-3)^2 > 0$ for $x \ne 3$. Therefore, two functions are linearly independent in any interval.

6.4 (10 pts) The characteristic equation for a certain differential equation is given. State the order of the differential equation and give the form of the general solution.

$$(\mathbf{a})\ \ 2\,\lambda^3 - \lambda^2 - 7\,\lambda + 6 = 0; \qquad\qquad (\mathbf{b})\ \ 3\,\lambda^3 - 20\,\lambda^2 + 39\,\lambda = 18.$$

Solution:

(a) Since the characteristic polynomial $2\,\lambda^3 - \lambda^2 - 7\,\lambda + 6 = (\lambda+2)(2\lambda-3)(\lambda-1)$ is of order three, the general solution becomes

$$y = a\,e^{-2t} + b\,e^{3t/2} + c\,e^{t}.$$

(b) Since the characteristic polynomial $3\,\lambda^3 - 20\,\lambda^2 + 39\,\lambda - 18 = (3\lambda - 2)(\lambda - 3)^2$ is of order three, the general solution becomes

$$y = a\,e^{2t/3} + bt\,e^{3t}.$$

6.5 (20 pts) Find the solution of the given initial value problem.

(a) $6\ddot{y} + \dot{y} - y = 0, \quad y(0) = 1,\ \dot{y}(0) = 2.$
(b) $\ddot{y} - 3\dot{y} = 0, \quad y(0) = 1,\ \dot{y}(0) = 3.$

Solution:

(a)

$$y = 3\,e^{t/3} - 2\,e^{-t/2}.$$

(b)

$$y = \frac{1}{3} + \frac{2}{3}\,e^{3t}.$$

6.6 (20 pts) Find the form of a particular solution $y_p(t)$ to the following ODEs to be used in the method of undetermined coefficients. **Do not solve for the coefficients!**

(a) $\ddot{y} - 4\,\dot{y} + 4\,y = 3t\,e^{2t},$ (b) $\ddot{y} + 4\,\dot{y} + 13\,y = 4\,e^{-2t}\sin 3t,$
(c) $\ddot{y} + \dot{y} - 2\,y = e^{-2t} + t\,e^{t} + t,$ (d) $\dddot{y} - 5\,\ddot{y} + 7\,\dot{y} - 3\,y = e^{-3t} + t\,e^{t}.$

Solution:

(a) The characteristic polynomial $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$ has one double root $\lambda = 2$. Therefore, we seek a particular solution in the form

$$y = t^2\,(b_0 + b_1 t)\,e^{2t}.$$

(b) The characteristic polynomial $\lambda^2 + 4\lambda + 13 = (\lambda + 2)^2 + 9$ has two complex conjugate roots $\lambda = -2 \pm 3\mathbf{j}$, so we seek a particular solution in the form

$$y = at\,e^{-2t}\cos 3t + bt\,e^{-2t}\sin 3t.$$

(c) The characteristic polynomial of the corresponding homogeneous equation is $\lambda^2 + \lambda - 2 = (\lambda + 2)\,(\lambda - 1)$. The control numbers of the forcing term are $\mu = -2$, $\mu = 1$, and $\mu = 0$. Therefore, we seek a particular solution in the form

$$y = ta\,e^{-2t} + t\,(b_0 + b_1 t)\,e^{t} + c_0 + c_1 t.$$

(d) Since the characteristic equation $(\lambda - 1)^2\,(\lambda - 3) = 0$ has two real roots one of which $(\lambda = 1)$ matches control number, we seek a particular solution in the form

$$y = a\,e^{-3t} + t^2\,(b_0 + b_1 t)\,e^{t}.$$

6.7 **[10 pts.]** For given family of solutions $c_1x^2 + c_2e^{-3x}\cos 5x$ to a constant coefficient differential equation $L[\mathtt{D}]\,y = 0$, find a linear differential operator of least possible order that annihilates the family.

Solution:

$$\mathtt{D}^3\left[(\mathtt{D}+3)^2 + 25\right] y = 0.$$

6.8 **[10 pts.]** Let $\mathtt{D}$ stand for the derivative operator. Write the general solution of the following differential equation

$$(\mathtt{D}-2)^3\left[(\mathtt{D}+5)^2 + 9\right]^2 y = 0.$$

Solution:

$$y = \left(a_0 + a_1t + a_2t^2\right)e^{2t} + (b_0 + b_1t)\,e^{-5t}\cos 3t + (c_0 + c_1t)\,e^{-5t}\sin 3t.$$