APMA 0330 — Applied Mathematics - I

Brown University **Fall, 2017**
Solutions to Homework, Set 5 **Due November 1**

5.1 (30 pts) Consider the initial value problem

$$3\,y' + y = x, \qquad y(0) = -2.$$

Upon introducing uniform grid $x_n = n\,h$ ($n = 0, 1, 2, \ldots$) with step size $h = 0.1$, find an approximate value of $y(1)$ using

- Euler's rule $y_{n+1} = y_n + h\, f(x_n, y_n)$, and
- Heun's method $y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n=1}, y_n + h\, f(x_n, y_n))\right]$.

Compare your answer with the true value $\phi(1)$, where $y = \phi(t)$ is the actual solution.

Solution: The given initial value problem has the solution

$$y = \phi(x) = e^{-x/3} - 3 + x.$$

Therefore, $\phi(1) = -1.2834686894262108$.

- Using Euler's rule $y_{n+1} = y_n + h\, f(x_n, y_n)$ for the slope function $f(x, y) = (x - y)/3$, we get

$$y_{n+1} = y_n + \frac{h}{3}\left(x_n - y_n\right), \qquad y_0 = -2.$$

We use *Mathematica* to generate the sequence of Euler points:

```
x0 = 0; y0 = -2; h = 0.1;
Y[0] = y0; X[0] = x0;
f[x_, y_] = (x - y)/3
For[i = 1, i <= 10, i++, X[i] = x0 + i*h;
  Y[i] = Y[i - 1] + f[X[i - 1], Y[i - 1]]*h;];
data = Table[{X[i], Y[i]}, {i, 0, 10}]
{{0, -2}, {0.1, -1.93333}, {0.2, -1.86556}, {0.3, -1.7967}, {0.4, \
-1.72681}, {0.5, -1.65592}, {0.6, -1.58406}, {0.7, -1.51125}, {0.8, \
-1.43755}, {0.9, -1.36296}, {1., -1.28753}}
```

- Using Heun's method $y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n=1}, y_n + h\, f(x_n, y_n))\right]$ for $f(x, y) = (x - y)/3$, we get

$$y_{n+1} = y_n + \frac{h}{6}\left(x_n - y_n\right), \qquad y_0 = -2.$$

We use *Mathematica* to generate the sequence of Heun points:

```
f[x_, y_] = (x - y)/3;
y[0]=-2; h = 0.1;
Do[k1 = h f[h n, y[n]];
 k2 = h f[h (n + 1), y[n] + k1];
 y[n + 1] = y[n] + .5 (k1 + k2), {n, 0, 9}]
y[10]
Out[8]= -1.28342
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So we see that the Heun method gives the better approximation than the Euler method.

5.2 **(10 pts)** Find the separatrix to the differential equation $y' = 4\,\sin(2t) + 2\,y - 8$.

Solution: Since the general solution of the given differential equation is $y = 4 - \cos 2t - \sin 2t + C\,e^{2t}$, the separatrix will be

$$4 - \cos 2t - \sin 2t.$$

5.3 **(20 pts)** Solve the equations with the dependent variable missing.

(a) $xy''' - 2y'' = 0$; **(b)** $y''' - y'' = 1$;
(c) $x^2\,y'' + 2y' = 4x$; **(d)** $y'' + y' = 4\,\sinh x$.

Solution:

(a) We set $p = y''$ and reduce the given differential equation to a first order equation:

$$x\,p' - 2p = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}p}{p} = 2\,\frac{\mathrm{d}x}{x}.$$

Integration yields

$$p = x^2 + C_1 \qquad \Longrightarrow \qquad y'' = p = x^2 + C_1.$$

We integrate twice to obtain the general solution:

$$y = \frac{x^4}{12} + C_1 x^2 + C_2 x + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

(b) Setting $p = y''$, we get the linear differential equation, which is also a separable one

$$p' - p = 1 \qquad \Longrightarrow \qquad \frac{\mathrm{d}p}{p+1} = \mathrm{d}x.$$

Integration yields

$$p = C_1 e^x - 1 \qquad \Longrightarrow \qquad y'' = p = C_1 e^x - 1.$$

Integrating again, we obtain the general solution

$$y = C_1 e^x - \frac{x^2}{2} + C_2 x + C_3.$$

(c) Setting $p = y'$, we obtain the first order differential equation

$$x^2 p' + 2p = 4x \qquad \Longrightarrow \qquad p' + \frac{2}{x^2}\,p = \frac{4}{x}.$$

An integrating factor is obtained from the differential equation

$$\mu' - \frac{2}{x^2}\,\mu = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}\mu}{\mu} = 2\,\frac{\mathrm{d}x}{x^2} \qquad \Longrightarrow \qquad \mu = e^{-2/x}.$$

Multiplying by μ, we obtain an exact equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-2/x}p\right] = \frac{4}{x}\,e^{-2/x} \qquad \Longrightarrow \qquad e^{-2/x}p = \int \frac{4}{x}\,e^{-2/x}\mathrm{d}x + C_1.$$

Next integrating gives the required solution:

$$y(x) = \int \mathrm{d}x e^{2/x} \int \frac{4}{x}\,e^{-2/x}\mathrm{d}x + C_1 \int \mathrm{d}x\, e^{2/x} + C_2.$$

(d) We set $p = y'$, then $p' = y''$ and the equation becomes

$$p' + p = 4\,\sinh x,$$

which is a linear equation. Using the Bernoulli method, we seek its solution as the product $p(x) = u(x)\,v(x)$, where

$$u' + u = 0 \qquad \Longrightarrow \qquad u = e^{-x},$$

$$u\,v' = 4\,\sinh x \qquad \Longleftrightarrow \qquad v' = 3\left(e^{2x} - 1\right) \qquad \Longrightarrow \qquad v = e^{2x} - 2x + C,$$

where C is a constant of integration. Therefore, the general solution becomes

$$p = u\,v = e^{x} - 2x\,e^{-x} + C\,e^{-x} \qquad \Longrightarrow \qquad y = e^{x} + 2\,e^{-x} + 2x\,e^{-x} - C\,e^{-x} + C_2.$$

5.4 (20 pts) Solve the equations with the independent variable missing.

(a) $y'' + 2y(y')^3 = 0$; **(b)** $2yy'' = y^2 + (y')^2$;
(c) $y\,y'' = (y')^3$; **(d)** $y'' + (y')^2 = 2\,e^{-y}$.

Solution: In all problems we use the chain rule:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}y}\,\frac{\mathrm{d}y}{\mathrm{d}x} = y'\,\frac{\mathrm{d}v}{\mathrm{d}y}.$$

We also set $v = y'$

(a) We have

$$v\,\frac{\mathrm{d}v}{\mathrm{d}y} + 2y\,v^3 = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}v}{\mathrm{d}y} = -2yv^2.$$

Separating variables, we get

$$-\frac{\mathrm{d}v}{v^2} = 2y\,\mathrm{d}y \qquad \Longrightarrow \qquad \frac{1}{v} = y^2 + C_1,$$

where C_1 is an arbitrary constant. Next separation of variables yields

$$v = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y^2 + C_1} \qquad \Longrightarrow \qquad \left(y^2 + C_1\right)\mathrm{d}y = \mathrm{d}x.$$

Integration gives the solution (in implicit form):

$$\frac{y^3}{3} + C_1 y = x + C_2.$$

(b) We have

$$2yv\,\frac{\mathrm{d}v}{\mathrm{d}y} = y^2 + v^2 \qquad \Longrightarrow \qquad \frac{\mathrm{d}v}{\mathrm{d}y} = \frac{y^2 + v^2}{2yv}.$$

Setting $v = y\,z(y)$, we reduce our equation in v to a separable one:

$$\frac{\mathrm{d}v}{\mathrm{d}y} = z + y\,z' = \frac{1 + z^2}{2z} \qquad \Longrightarrow \qquad \frac{2z}{1 - z^2}\,\mathrm{d}z = \frac{\mathrm{d}y}{y}.$$

Integration yields

$$-\ln\left|1 - z^2\right| = \ln C_1 y \qquad \Longrightarrow \qquad 1 - z^2 = \frac{C_1}{y} \qquad \Longrightarrow \qquad z^2 = 1 - \frac{C_1}{y}.$$

Since $z = v/y$, we get

$$v^2 = y^2 - C_1 y \qquad \Longrightarrow \qquad v = \frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{y^2 - C_1 y}$$

Again, separating variables, we obtain

$$\frac{\mathrm{d}y}{\sqrt{y^2 - C_1 y}} = \mathrm{d}x \qquad \Longrightarrow \qquad 2\,\ln\left[\sqrt{y} + \sqrt{y - C_1}\right] = x + C_2.$$

(c) We have

$$yv\,\frac{\mathrm{d}v}{\mathrm{d}y} = v^3 \qquad \Longrightarrow \qquad \frac{\mathrm{d}v}{\mathrm{d}y} = \frac{v^2}{y}.$$

Separation of variables gives

$$\frac{\mathrm{d}v}{v^2} = \frac{\mathrm{d}y}{y} \qquad \Longrightarrow \qquad -\frac{1}{v} = \ln C_1 y \qquad \Longrightarrow \qquad v = -\frac{1}{\ln C_1 y},$$

where C_1 is a constant. Next separation of variables yields

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v = -\frac{1}{\ln C_1 y} \qquad \Longrightarrow \qquad (\ln C_1 y)\,\mathrm{d}y = -\mathrm{d}x.$$

Integrating both sides, we obtain the general solution (in implicit form):

$$y\,\ln C_1 y - y = -x + C_2.$$

(d) We have

$$v\,\frac{\mathrm{d}v}{\mathrm{d}y} + v^2 = e^{-y} \qquad \Longrightarrow \qquad \frac{\mathrm{d}v}{\mathrm{d}y} + v = v^{-1} e^{-y}.$$

Since the equation in v is a Bernoulli equation, we seek its solution in the form: $v = u\,w$, where u is a solution of the linear part:

$$u' + u = 0 \qquad \Longrightarrow \qquad u = e^{-y}.$$

The function w is the general solution of the separable equation:

$$u\,w' = \frac{1}{u\,w}\,e^{-y} = \frac{1}{w} \qquad \Longrightarrow \qquad w\,w' = e^{y}.$$

Integration yields

$$\frac{w^2}{2} = e^y + C_1 \qquad \Longrightarrow \qquad w = \pm\sqrt{2\,e^y + C_1}.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v = u\,w = e^{-y}\sqrt{2\,e^y + C_1} \qquad \Longrightarrow \qquad \frac{\mathrm{d}y}{e^{-y}\sqrt{2\,e^y + C_1}} = \mathrm{d}x.$$

Next integration yields the general solution (in implicit form)

$$\int \frac{\mathrm{d}y}{e^{-y}\sqrt{2\,e^y + C_1}} = \sqrt{2\,e^y + C_1} = x + C_2.$$

Squaring both sides, we get

$$2\,e^y + C_1 = (x + C_2)^2\,.$$

5.5 **(10 pts)** Determine the longest interval in which the given initial value problem is certain to have a unique twice-differentiable solution.

(a) $(t-1)\,y'' - 4t\,y' + 5\,y = \cos t, \qquad y(0) = 2, \quad y'(0) = 1;$

(b) $t(t^2-4)\,y'' + t\,y' + (\ln|t|)\,y = 0, \qquad y(1) = 0, \quad y'(1) = 2.$

Solution:

(a) We rewrite the given differential equation in normal form:

$$y'' - \frac{4t}{t-1} + \frac{5}{t-1}\, y = \frac{\cos t}{t-1}.$$

The coefficients of the above equation are undefined at $t = 1$. Therefore, the validity interval becomes

$$-\infty < t < 1.$$

(b) We rewrite the given differential equation in normal form:

$$y'' + \frac{1}{t^2-4}\, y' + \frac{\ln|t|}{t(t^2-4)}\, y = 0.$$

The coefficients of the above equation are undefined at $t = \pm 2$ and $t = 0$ (where logarithm blows up). Therefore, the validity interval becomes

$$0 < t < 2.$$

5.6 (10 pts) Find the Wronskian of two solutions of the differential equation $t^2 y'' - t(t+2)\, y' + (t+2)\, y = 0$ without solving the equation.

Solution: The Wronskian of any two solutions to the differential equation $y'' + p(x)\, y' + q(x)\, y = 0$ is

$$W = C\, \exp\left\{ - \int \mathrm{d}x\, p(x) \right\},$$

where C is a constant. In our case, $p(t) = -\frac{t+2}{t} = -1 - \frac{2}{t}$, so the Wronskian will be

$$W = C\, t^2\, e^t.$$