APMA 0330 — Applied Mathematics - I

Brown University **Fall, 2017**

Solutions to Homework, Set 4 **Due October 11**

4.1 (14 pts) Determine the validity interval for each of the following initial value problems.

- 3 pts all intervals where both $a(x)$ and $f(x)$ are continuous.
- 3 pts validity interval.

(a) $(x^2 - 4)\, y' + x^5 y = x^2 + 1, \quad y(0) = 1;$

(b) $(\cos \pi x)\, y' + (\sin x)\, y = \tan \pi x, \quad y(1) = 1.$

Solution: In all problems, we reduce the given differential equation to the standard form:

$$y' + a(x)\, y = f(x).$$

(a) With $a(x) = \dfrac{x^5}{x^2-4} = \dfrac{x^5}{(x-2)(x+2)}$ and $f(x) = \dfrac{x^2+1}{(x-2)(x+2)}$, we see that both functions are continuous everywhere except points $x = \pm 2$. Therefore, the validity interval including the initial point $x = 0$ will be $\boxed{-2 < x < 2}$

(b) With $a(x) = \sin x / \cos \pi x$ and $f(x) = \tan \pi x \csc \pi x = \sin \pi x/(\cos \pi x)^2$, we see that the function $a(x)$ has points of discontinuity at $x = 1/2 + n$, $n = 0, \pm 1, \pm 2, \ldots$. The forcing function $f(x)$ is not defined at the same points. Therefore, the validity interval will be $\boxed{1/2 < x < 3/2}$

4.2 (6 pts) An inductor-resistor series circuit (LR circuit) can be modeled by the following differential equation (the initial condition is assumed to be given $i(0) = i_0 = 0.5$):

$$V_t = V_R(t) + V_L(t) \qquad \Longrightarrow \qquad L\,\frac{\mathrm{d}i}{\mathrm{d}t} + R\, i = V(t) = \begin{cases} 6, & \text{for } 0 < t < 2\,\tau, \\ 0, & \text{otherwise,} \end{cases}$$

where the voltage drop across the resistor is $V_R = i\, R$ (Ohms Law), $R = 1$ being in Ohms, the voltage drop across the inductor is $V_L = L\, \mathrm{d}i/\mathrm{d}t$, $L = 0.1$ being in Henries. The $\tau = L/R$ term in the above equation is known commonly as the time constant. Plot the solution to IVP.

- 2 pts Solve the initial value problem
- 2 pts Write the final solution
- 2 pts Plot the solution

Solution: Solving the initial value problem

$$\frac{\mathrm{d}i}{\mathrm{d}t} + 10\, i = \begin{cases} 60, & \text{for } 0 < t < 0.2, \\ 0, & \text{otherwise,} \end{cases} \qquad\qquad i(0) = 0.5,$$

we get

$$i(t) = \frac{1}{2}\,e^{-10\,t} \times \begin{cases} 12\,e^{10\,t} - 11, & \text{for } 0 \leqslant t < 0.2, \\ 12\,e^{2} - 11, & 0.2 < t. \end{cases}$$

We check the answer with *Mathematica* and plot the solution.

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V[t_] = Piecewise[{{60, 0 < t < 2/10}}]
s = DSolve[{q'[t] + q[t]*10 == V[t], q[0] == 1/2}, q[t], t]
Plot[Evaluate[q[t] /. s], {t, 0, 0.6}, PlotStyle -> {Blue, Thick}]
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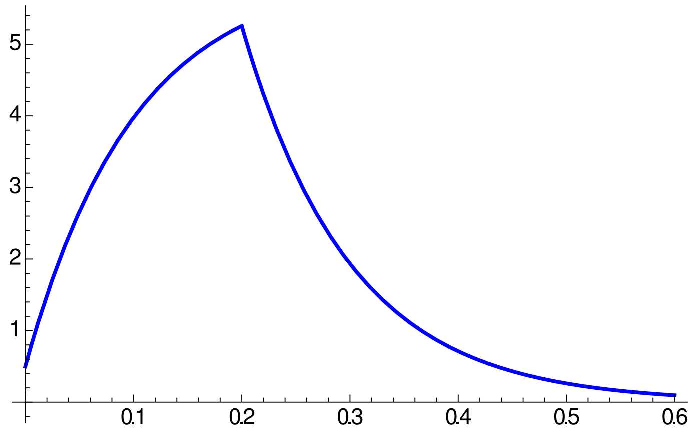

Figure 1: Solution to problem 2, plotted with Mathematica.

Partial Credits for Problem 3-6

5 points for problem 3 and 5.

7 points for problem 4 and 6. 2 additional points for substituting initial value to calculate the constant C.

Integrating factor method

- **1 pt** Formula or differential equation of integrating factor.
- **1 pt** Solve integrating factor correctly
- **1 pt** Exact Equation
- **1 pt** Integration
- **1 pt** Final solution

Bernoulli method

- **1 pt** Differential equation for u
- **1 pt** Find u correctly

- **1 pt** Differential equation for v
- **1 pt** Integration
- **1 pt** Final solution

4.3 (20 pts) Solve the linear equations.

(a) $\dfrac{dy}{dx} = \dfrac{\sin x + (2y - x)\cos x}{\sin x}$; **(b)** $\dfrac{dy}{dx} + y = \dfrac{1}{1+e^{-x}}$;

(c) $\dfrac{dy}{dx} + y\,\sin x = 2x\,\sin x$; **(d)** $\dfrac{dy}{dx} - y\,\ln x = x^2$.

Solution: **(a)** Solving equation for an integrating factor

$$\mu'(x) + 2\mu\,\cot x = 0 \qquad \Longrightarrow \qquad \int \frac{d\mu}{\mu} = \ln\mu = -2\int \cot x\,dx = -2\ln\sin x,$$

we find $\mu(x) = \sin^{-2} x$. Multiplying by $\mu(x)$, we get an exact equation:

$$\frac{d}{dx}\left[y(x)\,\sin^{-2} x\right] = \sin^{-2} x\,(1 - x\,\cos x)\,.$$

Upon integration, we obtain

$$y(x)\,\sin^{-2} x = \int \sin^{-2} x\,[1 - x\,\cos x]\,dx = \frac{1}{2}\left(x\,\csc^2 x - \cot x\right) + C,$$

where C is a constant of integration. Multiplication by $\sin^2 x$ gives the general solution

$$y(x) = C\,\sin^2 x + \frac{1}{2}\,(x - \sin x\,\cos x)\,.$$

(b) Using the Bernoulli method, we seek its solution as a product $y = u\,v$, where u is a solution of the separable equation $u' + u = 0$, which gives $u(x) = e^{-x}$. For $v(x)$, we also have a separable equation

$$u\,v' = \frac{1}{1+e^{-x}} \qquad \Longrightarrow \qquad v = \int \frac{e^x}{1+e^{-x}}\,dx = e^x - \ln(1+e^x) + C.$$

Therefore, the general solution becomes

$$y(x) = C\,e^{-x} + e^{-x}\left(e^x - \ln(1+e^x)\right) = C\,e^{-x} + 1 - e^{-x}\,\ln(1+e^x)\,.$$

(c) Using the Bernoulli method, we seek its solution as a product $y = u\,v$, where u is a solution of the separable equation $u' + u\,\sin x = 0$, which gives $u(x) = e^{\cos x}$. For $v(x)$ we also have a separable equation

$$u\,v' = 2x\,\sin x \qquad \Longrightarrow \qquad v = \int 2x\,\sin x\,e^{\cos x}\,dx + C.$$

Then the general solution becomes

$$y(x) = C\,e^{-\cos x} + e^{-\cos x} \int 2x\,\sin x\,e^{\cos x}\,\mathrm{d}x.$$

(d) Solving an equation for an integrating factor $\mu' + \mu\,\ln x = 0$, we get

$$\ln \mu(x) = x\,(1 - \ln x) \qquad \Longrightarrow \qquad \mu(x) = e^{x - x\,\ln x} = e^{x}\,e^{-x\,\ln x}.$$

Multiplication by $\mu(x)$ reduces the given equation to an exact equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y(x)\,\mu(x)\right] = x^3\,\mu(x).$$

Integration yields the general solution:

$$y(x) = \frac{1}{\mu(x)}\left[\int x^3\,\mu(x)\,\mathrm{d}x + C\right] = e^{-x}\,e^{x\,\ln x}\left[\int x^3\,e^{x}\,e^{-x\,\ln x}\,\mathrm{d}x + C\right].$$

4.4 (20 pts) Find the particular solution to the given initial value problem.

(a) $x\,y' + (x+2)\,y = 2\,\sin x, \qquad y(\pi) = -1;$

(b) $x^2 y' - 4x\,y = x^4, \qquad y(1) = 2;$

(c) $y' + 3y = f(x) = \begin{cases} 9x, & \text{if } 0 \le x < 1, \\ 9, & \text{if } 1 \le x < \infty; \end{cases}, \qquad y(0) = 0;$

(d) $x^2 y' + 2x\,y = \cos x, \qquad y(\pi) = 0.$

Solution: **(a)** Using the Bernoulli method, we seek its solution as a product $y = u\,v$, where u is a solution of the separable equation $x\,u' + u\,(x+2) = 0$, which gives $u(x) = x^{-2}e^{-x}$. For $v(x)$ we also have a separable equation

$$x\,u\,v' = 2\,\sin x \qquad \Longrightarrow \qquad v' = x\,e^{x}\,2\,\sin x.$$

Integration yields

$$v(x) = e^{x}\left[(1-x)\cos x + x\,\sin x\right] + C.$$

Multiplying by $u(x) = x^{-2}e^{-x}$, we get the general solution:

$$y(x) = C\,x^{-2}e^{-x} + x^{-2}\,(1-x)\cos x + x^{-1}\sin x.$$

Setting $x = \pi$ and equating the result to -1, we obtain

$$C = e^{\pi}\left(1 - \pi - \pi^2\right).$$

(b) Using the Bernoulli method, we seek its solution as a product $y = u\,v$, where u is a solution of the separable equation

$$\frac{\mathrm{d}u}{u} = \frac{4x}{x^2}\,\mathrm{d}x \qquad \Longrightarrow \qquad u = x^4.$$

For $v(x)$ we also have a separable equation

$$x^2 u\,v' = x^4 \qquad \Longrightarrow \qquad v' = 1/x^2 \qquad \Longrightarrow \qquad v = C - x^{-1}.$$

This gives us the general solution:

$$y = u(x)\,v(x) = C\,x^4 - x^3.$$

From the initial condition, we get $C = 3$.

(c) First, we solve the initial value problem in the interval $0 \le x < 1$:

$$y' + 9y = 9x, \qquad y(0) = 0.$$

Multiplying both sides by an integrating factor $\mu(x) = e^{3x}$, we get an exact equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{3x}y(x)\right] = 9x\,e^{3x} \qquad \Longrightarrow \qquad e^{3x}y(x) = e^{3x}\,(3x-1) + C.$$

To satisfy the initial condition, we set C to be 1, so we have

$$y(x) = 3x - 1 + e^{-3x} \quad \text{for } 0 \le x \le 1.$$

Setting $x = 1$, we get $y(1) = 2 - e^{-3}$; therefore, we have to solve the following initial value problem for $1 < x$:

$$y' + 3y = 9, \qquad y(1) = 2 - e^{-3}.$$

Its solution becomes

$$y(x) = \begin{cases} e^{-3x} - 1 + 3x, & \text{for } 0 < x \le 1, \\ e^{-3x}\,(1 - e^3) + 3, & \text{for } 1 \le x < \infty. \end{cases}$$

(d) We solve the given differential equation using Bernoulli method: $y = u\,v$, where u is a solution of the homogeneous equation

$$x^2 u' + 2x\,u = 0 \qquad \Longrightarrow \qquad u(x) = x^{-2}.$$

Then for v we get the following separable equation:

$$x^2 u\,v' = \cos x \qquad \Longrightarrow \qquad v' = \cos x \qquad \Longrightarrow \qquad v = \sin x + C,$$

where C is an arbitrary constant. Multiplying $v(x)$ by $u = x^{-2}$, we get the general solution:

$$y = x^{-2}\,\sin x + C\,x^{-2}.$$

From the initial condition $y(\pi) = 0$, we obtain $C = 0$. Hence,

$$y = x^{-2}\,\sin x.$$

4.5 (20 pts) Solve the following Bernoulli equations.

(a) $x\,y' - y = -3\,x^4 y^3$; **(b)** $x\,y' = (x+1)\,y - 2\,y^3$;
(c) $3y' + 2y^4 x\,e^{-3x} = y$; **(d)** $y' + 2y\,\csc(2x) = y^2$.

Solution: **(a)** Using the Bernoulli method, we seek its solution as a product $y = u\,v$, where u is a solution of the separable equation $x\,u' - u = 0$, which gives $u = x$. Then for $v(x)$ we have a separable equation:

$$x\,u\,v' = -3\,x^4 u^3 v^3 \qquad \Longrightarrow \qquad -\frac{\mathrm{d}v}{v^3} = 3\,x^5 \mathrm{d}x.$$

Integration yields

$$\frac{1}{2\,v^2} = 3\,\frac{x^6}{6} + C \qquad \Longrightarrow \qquad v(x) = \left(x^6 + C\right)^{-1/2}.$$

Multiplying by $u(x)$, we get the general solution:

$$y(x) = x\left(x^6 + C\right)^{-1/2}.$$

(b) We use the Bernoulli method; so we seek the solution as the product of two functions $y(x) = u(x)\,v(x)$, where $u(x)$ is a solution of the "linear truncated" part:

$$x\,u' = (x+1)\,u \qquad \Longrightarrow \qquad \frac{\mathrm{d}u}{u} = \frac{x+1}{x}\,\mathrm{d}x.$$

Integration yields $u = x\,e^x$. Then for $v(x)$ we have a separable equation:

$$x\,u\,v' = -2\,u^3 v^3 \qquad \Longrightarrow \qquad -\frac{\mathrm{d}v}{v^3} = 2\,x\,e^{2x}\,\mathrm{d}x.$$

Integration yields

$$\frac{1}{v^2} = e^{2x}\,(2x-1) + C.$$

Therefore, the general solution becomes

$$y(x) = u(x)\,v(x) = x\,e^x\left[e^{2x}\,(2x-1) + C\right]^{-1/2}.$$

(c) We use the Bernoulli method; so we seek the solution as the product of two functions $y(x) = u(x)\,v(x)$, where $u(x)$ is a solution of the "linear truncated" part:

$$u' = u \qquad \Longrightarrow \qquad u = e^x.$$

Then for $v(x)$, we have a separable equation:

$$3u\,v' + 2u^4 v^4 x\,e^{-3x} = 0 \qquad \Longrightarrow \qquad -3\,\frac{\mathrm{d}v}{v^4} = 2x\,\mathrm{d}x.$$

Integration yields

$$\frac{1}{v^3} = x^2 + C \qquad \Longrightarrow \qquad v(x) = \left(x^2 + C\right)^{-1/3}.$$

We obtain the general solution upon multiplication by $u(x)$:

$$y(x) = u(x)\,v(x) = e^x \left(x^2 + C\right)^{-1/3}.$$

(d) We use the Bernoulli method; so we seek the solution as the product of two functions $y(x) = u(x)\,v(x)$, where $u(x)$ is a solution of the "linear truncated" part:

$$u' + 2u\ \csc x = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}u}{u} = -2\ \csc 2x\,\mathrm{d}x.$$

Integration yields

$$\ln u = -\ln\frac{\sin x}{\cos x} = \ln\frac{\cos x}{\sin x} \qquad \Longrightarrow \qquad u(x) = \frac{\cos x}{\sin x} = \cot x.$$

Substituting the product $y = u\,v$ into the given equation, we get a separable equation for v:

$$u\,v' = u^2 v^2 \qquad \Longrightarrow \qquad \frac{\mathrm{d}v}{v^2} = u\,\mathrm{d}x \qquad \Longrightarrow \qquad -\frac{1}{v} = \ln|\sin x| - C.$$

Therefore the general solution becomes

$$y(x) = \frac{\cot x}{C - \ln|\sin x|}.$$

4.6 (20 pts) Solve the initial value problems for the Bernoulli equation.

(a) $x\,y' + y = x^4 y^3, \qquad y(1) = 1/4;$

(b) $x\,y' + 3\,y = x^3 y^2, \qquad y(1) = 1/2.$

Solution: **(a)** First, we find the general solution using the Bernoulli method: $y = u\,v$, where u is a solution of the "linear truncated" part:

$$x\,u' + u = 0 \qquad \Longrightarrow \qquad u = x^{-1}.$$

Then for $v(x)$ we get a separable equation:

$$xu\,v' = x^4 u^3 v^3 \qquad \Longrightarrow \qquad \frac{\mathrm{d}v}{v^3} = x\,\mathrm{d}x \qquad \Longrightarrow \qquad -\frac{1}{2v^2} = \frac{x^2}{2} + C.$$

Hence, the general solution becomes

$$y(x) = u(x)\,v(x) = x^{-1}\left(C - x^2\right)^{-1/2} \qquad \Longrightarrow \qquad y(1) = (C-1)^{-1/2} = 1/4.$$

Therefore, $C = 3$ and we get

$$y(x) = x^{-1}\left(3 - x^2\right)^{-1/2}$$

(b) First, we find the general solution using the Bernoulli method: $y = u\,v$, where u is a solution of the "linear truncated" part:

$$x\,u' + 3u = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}u}{u} = -\frac{3}{x}\,\mathrm{d}x$$

Integrating, we obtain $u(x) = x^{-3}$. Then for $v(x)$, we get a separable equation:

$$x\,u\,v' = x^3 u^2 v^2 \qquad \Longrightarrow \qquad v' = x^{-1} v^2.$$

The general solution becomes

$$y = x^{-3}\left(C - \ln|x|\right)^{-1} \qquad \Longrightarrow \qquad C = 2.$$