APMA 0330 — Applied Mathematics - I

Brown University **Fall, 2017**

Solutions to Homework, Set 3 **Due October 4**

3.1 (12 pts) Given a potential function $\psi(x,y)$, find the exact differential equation $\mathrm{d}\psi(x,y) = 0$.

- [1pt] differentiate with respect to x and y.
- [1pt] show identity of partial derivatives and conclude that ϕ is a potential function.
- [1pt] final solution.

(a) $\psi(x,y) = 3\,x^2 + 5\,y^2$; **(b)** $\psi(x,y) = \exp(3\,x^2 y^3)$;
(c) $\psi(x,y) = \ln(x^3 y^4)$; **(d)** $\psi(x,y) = (2x + 3y - 5)^2$.

Solution: In all problems, we use the definition of the differential:

$$\mathrm{d}\psi(x,y) = \frac{\partial \psi}{\partial x}\,\mathrm{d}x + \frac{\partial \psi}{\partial y}\,\mathrm{d}y.$$

(a) $6\,x\,\mathrm{d}x + 10\,y\,\mathrm{d}y = 0$;

(b) $6x\,y^3 e^{3\,x^2 y^3}\,\mathrm{d}x + 9x^2 y^2\,e^{3\,x^2 y^3}\,\mathrm{d}y = 0$; (can multiple $e^{3\,x^2 y^3}$ be dropped ?)

(c) $\frac{3}{x}\,\mathrm{d}x + \frac{4}{y}\,\mathrm{d}x = 0$; (can multiple 2 be dropped ?)

(d) $2 \cdot (2x + 3y - 5)\,[2\,\mathrm{d}x + 3\,\mathrm{d}y] = 0$. (can multiple $(2x + 3y - 5)$ be dropped ?)

3.2 (20 pts) Show that the following differential equations are exact and solve them

(a) $3x^2 y^2\,y' + 2y^3 x = 0$; **(b)** $y\,(e^{xy} + y)\,\mathrm{d}x + x\,(e^{xy} + 2y)\,\mathrm{d}y = 0$;
(c) $(3x^2 y + 2x\,e^y)\,\mathrm{d}x + (x^2 e^y + x^3)\,\mathrm{d}y = 0$; **(d)** $(2xy^2 - 3)\,\mathrm{d}x + (2x^2 y + y^2)\,\mathrm{d}y = 0$.

Solution: In all problems, we use the following condition for exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{or in short} \qquad M_y = N_x.$$

Then there exists a potential function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial \psi}{\partial y} = N(x,y). \tag{1}$$

- [1pt] differentiate with respect to x and y
- [1pt] show identity of partial derivatives
- [2pts] Integrate M with respect to x (or integrate N with respect to y), 1 point is deducted if the integral does not contain arbitrary function of y.
- [1pt] Final result.

(a) With $N = 3x^2y^2$ and $M = 2xy^3$, we have $M_y = 6xy^2$ and $N_x = 6xy^2$. So the given differential equation is exact. Then equations (1) becomes:

$$\psi_x = 2xy^3 \qquad \text{and} \qquad \psi_y = 3x^2y^2.$$

Integrating the former ($\psi_x = M$), we get $\psi(x,y) = x^2y^3 + h(y)$, where $h(y)$ is an arbitrary function in variable y. Substituting this form of $\psi(x,y)$ into the latter ($\psi_y = N$), we obtain

$$\psi_y = 3x^2y^2 + h'(y) = N(x,y) = 3x^2y^2 \qquad \Longrightarrow \qquad h'(y) = 0.$$

Therefore, $h(y)$ is a constant that we drop and get the general solution to be

$$\psi(x,y) \equiv x^2y^3 = C.$$

(b) With $M = y\,e^{xy} + y^2$ and $N = x\,e^{xy} + 2xy$, we have $M_y = e^{xy} + xy\,e^{xy} + 2y$ and $N_x = e^{xy} + xy\,e^{xy} + 2y$. Therefore, the given differential equation is exact, and we have two equations for a potential function:

$$\psi_x = y\,e^{xy} + y^2 \qquad \text{and} \qquad \psi_y = x\,e^{xy} + 2xy.$$

Integration of the latter gives $\psi(x,y) = e^{xy} + x\,y^2 + h(y)$. Then we use the former to obtain

$$\psi_y = x\,e^{xy} + 2x\,y + h'(y) = N = x\,e^{xy} + 2xy \qquad \Longrightarrow \qquad h' = 0.$$

So we get the general solution:

$$\psi(x,y) = e^{xy} + x\,y^2 = C.$$

(c) With $M = 3x^2y + 2x\,e^y$ and $N = x^2e^y + x^3$, we have $M_y = 3x^2 + 2x\,e^y = N_x$. Then equations (1) become

$$\psi_x = M = 3x^2y + 2x\,e^y \qquad \text{and} \qquad \psi_y = N = x^2\,e^y + x^3.$$

Integrating the latter, we get $\psi(x,y) = x^2\,e^y + x^3y + k(x)$, where $k(x)$ is arbitrary function of x. Substitution into the equation $\psi_x = M$ yields $3x^2y + 2x\,e^y = 2x\,e^y + 3x^2y + k'(x)$. Hence $k'(x) = 0$ and upon its integration, we get the general solution:

$$\psi(x,y) \equiv x^2\,e^y + x^3y = C.$$

(d) With $M = 2xy^2 - 3$ and $N = 2x^2y + y^2$, we have $M_y = 4xy = N_x$, so the given differential equation is exact. Then there exists a potential function $\psi(x,y)$, for which we know its partial derivatives according to (1):

$$\psi_x = M = 2xy^2 - 3 \qquad \text{and} \qquad \psi_y = N = 2x^2y + y^2.$$

Integrating the former, we get

$$\psi(x,y) = x^2y^2 - 3x + h(y) \qquad \Longrightarrow \qquad \psi_y = 2x^2y + h'(y) = 2x^2y + y^2.$$

Since $h'(y) = y^2$, we integrate it and obtain the general solution:

$$\psi(x,y) \equiv x^2y^2 - 3x + \frac{y^3}{3} = C.$$

3.3 (24 pts) Are the following equations exact? Solve the initial value problems.

(a) $\cos\pi x\,\cos 2\pi y\,\mathrm{d}x = 2\sin\pi x\,\sin 2\pi y\,\mathrm{d}y,\ \ y(3/2) = 1/3;$

(b) $2xy\,\mathrm{d}y + (x^2+y^2)\,\mathrm{d}x = 0,\ \ y(0) = 4;$

(c) $(3x^2y-5)\,\mathrm{d}x + (x^3+6y^2)\,\mathrm{d}y = 0,\ \ y(1) = 2;$

(d) $(\cos\theta - 2r\cos^2\theta)\,\mathrm{d}r + r\sin\theta(2r\cos\theta - 1)\,\mathrm{d}\theta = 0, \quad r(\pi/4) = 1.$

6 points for each problem. The same distribution as in Problem 2. Plus 1 point for substituting initial condition to compute the constant C.

Solution: In all problems, we first check for exactness and then find the potential function using the line integral:

$$\psi(x,y) = \int_{(x_0,y_0)}^{(x,y)} M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y, \tag{2}$$

where integration is conducted along some path connecting the given point (x_0,y_0) with an arbitrary point (x,y) taken along straight lines along coordinate axis.

(a) With $M = \sin\pi x\,\cos 3\pi y$ and $N = 3\cos\pi x\,\sin 3\pi y$, we have $M_y = -3\pi\,\sin\pi x\,\sin 3\pi y$ and $N_x = -3\pi\,\sin\pi x\,\sin 3\pi y$; therefore $M_y = N_x$ and the given differential equation is exact. To find a potential function, we use formula (2), where $x_0 = 3/2$ and $y_0 = 1/3$ and the path of integration is taken first horizontally, and then vertically:

$$\begin{aligned}
\psi(x,y) &= \int_{3/2}^{x} M(x,1/3)\,\mathrm{d}x + \int_{1/3}^{y} N(x,y)\,\mathrm{d}y \\
&= \int_{3/2}^{x} \sin\pi x\,\cos\pi\,\mathrm{d}x + \int_{1/3}^{y} 3\cos\pi x\,\sin 3\pi y\,\mathrm{d}y \\
&= -\int_{3/2}^{x} \sin\pi x\,\mathrm{d}x - 3\cos\pi x\int_{1/3}^{y}\sin 3\pi y\,\mathrm{d}y \\
&= \left[\frac{1}{\pi}\cos\pi x\right]_{x=3/2}^{x} - \cos\pi x\left[\frac{1}{\pi}\cos 3\pi y\right]_{y=1/3}^{y} \\
&= \frac{1}{\pi}\cos\pi x - \frac{2}{\pi}\cos\pi x\,\cos^2\frac{3\pi y}{2}.
\end{aligned}$$

Equating $\pi\,\psi(x,y)$ to zero, we obtain the solution (in implicit form):

$$\cos\pi x\left(1 - 2\,\cos^2\frac{3\pi y}{2}\right) = 0.$$

(b) With $M = 6xy$ and $N = 3x^2 + 4y^3$, we get $M_y = 6x = N_x$, so the given differential equation is exact and equations (1) must hold for some potential function $\psi(x,y)$. To find its explicit expression, we use line integral (2) with $x_0 = 3$ and $y_0 = 4$:

$$\psi(x,y) = \int_4^y \left(3x^2 + 4y^3\right)\mathrm{d}y = 3x^2\,(y-4) + y^4 - 4^4.$$

Upon simplification and equating $\psi(x,y)$ to zero, we obtain the required solution in implicit form:

$$y^4 + 3x^2y - 12x^2 - 256 = 0.$$

(c) With $M = 3x^2y - 5$ and $N = x^3 + 6y^2$, we get $M_y = 3x^2$ and $N_x = 3x^2$; therefore, the given differential equation is exact. Using line integral with $x_0 = 1$ and $y_0 = 2$, we obtain the potential function:

$$\psi(x,y) = \int_1^x \left(6x^2 - 5\right) \mathrm{d}x + \int_2^y \left(x^3 + 6y^2\right) \mathrm{d}y = 2x^3 - 2 - 5x + 5 + x^3(y-2) + 2y^3 - 2^4.$$

Upon simplification and equating $\psi(x,y)$ to zero, we obtain the required solution in implicit form:

$$x^3y - 5x + 2y^3 = 13.$$

(d) With $M(r,\theta) = \cos\theta - 2r\cos^2\theta$ and $N(r,\theta) = r\sin\theta(2r\cos\theta - 1)$, we have $M_\theta = -\sin\theta + 4r\,\cos\theta\,\sin\theta$ and $N_r = 4r\,\sin\theta\,\cos\theta - \sin\theta$. Therefore, the given differential equation is exact and we use formula (2) to obtain the potential function:

$$\begin{aligned}
\psi(r,\theta) &= \int_1^r M(r,\pi/4)\,\mathrm{d}r + \int_{\pi/4}^{\theta} N(r,\theta)\,\mathrm{d}\theta \\
&= \int_1^r \left(\frac{1}{\sqrt{2}} - r\right) \mathrm{d}r + \int_{\pi/4}^{\theta} r\sin\theta(2r\cos\theta - 1)\,\mathrm{d}\theta \\
&= \frac{r-1}{\sqrt{2}} - \frac{r^2}{2} + \frac{1}{2} - \frac{r}{\sqrt{2}} + r\,\cos\theta - \frac{r^2}{2}\,\cos 2\theta.
\end{aligned}$$

Using trigonometric identity $1 + \cos 2\theta = 2\,\cos^2\theta$, we simplify the potential function and equate it to zero:

$$\frac{1}{2} - \frac{1}{\sqrt{2}} + r\,\cos\theta - r^2\cos^2\theta = 0.$$

3.4 **(24 pts)** Show that the given equations are not exact, but become exact when multiplied by the corresponding integrating factor. Find an integrating factor as a function of x only and determine a potential function for the given differential equations.

6 points for each problem. The same distribution as in Problem 3.5 and plus 1 point for showing that equations are not exact.

(a) $y' + y(1+2x) = 0$; **(b)** $x^3\,y' = x^2y + 3x$;
(c) $(yx^3e^{xy} - 2y^3)\,\mathrm{d}x + (x^4e^{xy} + 3xy^2)\,\mathrm{d}y = 0$; **(d)** $4\,\mathrm{d}x - e^{y-2x}\,\mathrm{d}y = 0$.

Solution: In all problems, subscript M_x means the partial derivative $M_y = \dfrac{\partial M}{\partial y}$ and correspondingly $N_x = \dfrac{\partial N}{\partial x}$. Integrating factor as a function of y can be obtained explicitly:

$$\mu(x) = \exp\left\{ \int \frac{M_y - N_x}{N}\,\mathrm{d}x \right\}. \tag{3}$$

(a) With $M = y + xy$ and $N = 1$, we have $M_y = 1 + 2x$ and $N_x = 0$, so the given differential equation is not exact. However, the ratio $\dfrac{M_y - N_x}{N} = 1 + 2x$ is a function of x along. From equation (3), we find an integrating factor:

$$\mu(x) = \exp\left\{\int (1 + 2x)\,\mathrm{d}x\right\} = e^{x+x^2}.$$

Upon multiplication by $\mu(x)$, we get an exact equation with $M = (y + 2xy)\,e^{x+x^2}$ and $N = e^{x+x^2}$. Integrating $\psi_y = N$, we obtain

$$\psi(x, y) = y\,e^{x+x^2} + k(x),$$

where $k(x)$ is determined from the equation $\psi_x = M$, which becomes

$$\psi_x = y\,(1 + 2x)\,e^{x+x^2} + k'(x) = M = (y + 2xy)\,e^{x+x^2}.$$

Therefore, $k'(x) = 0$ and $k(x)$ is a constant, which we drop. The given problem has the general solution:

$$y\,e^{x+x^2} = C.$$

(b) With $M = x^2y + 3x$ and $N = -x^3$, we have $M_y = x^2$, $N_x = -3x^2$, so our differential equation is not exact. However, the ratios

$$\frac{M_y - N_x}{N} = \frac{x^2 + 3x^2}{-x^3} = -\frac{4}{x} \qquad \text{and} \qquad \frac{M_y - N_x}{M} = \frac{x^2 + 3x^2}{x^2y + 3x}$$

show that there exists an integrating factor as a function of x:

$$\mu(x) = x^{-4}.$$

Upon multiplication by $\mu(x)$, we get an exact equation:

$$\left(x^{-2}y + 3x^{-3}\right)\mathrm{d}x - x^{-1}\mathrm{d}y = 0.$$

Indeed,

$$\frac{\partial}{\partial y}\left(x^{-2}y + 3x^{-3}\right) x^{-2} \quad \text{and} \quad -\frac{\partial}{\partial x}\,x^{-1} = x^{-2}.$$

Therefore, there exists a potential function $\psi(x, y)$ such that $\psi_x = (x^{-2}y + 3x^{-3})$ and $\psi_y = -x^{-1}$. Integration of the latter yields

$$\psi(x, y) = -y\,x^{-1} + k(x) \qquad \Longrightarrow \qquad \psi_x = x^{-2}y + k'(x) = \left(x^{-2}y + 3x^{-3}\right).$$

Therefore, $k'(x) = 3x^{-3}$. This allows us to determine the general solution:

$$x^{-1}y + \frac{3}{2}\,x^{-2} = C \qquad \text{or} \qquad 2xy + 3 = C\,x^2.$$

(c) With $M = yx^3e^{xy} - 2y^3$ and $N = x^4e^{xy} + 3xy^2$, we have $M_y = x^3e^{xy} + yx^4e^{xy} - 6y^2$ and $N_x = (4x^3 + x^4y)\,e^{xy} + 3y^2$. The ratio

$$\frac{M_y - N_x}{N} = -3\,\frac{x^3e^{xy} + 3y^2}{x\,(x^3e^{xy} + 3y^2)} = -\frac{3}{x}$$

tells us that there exists an integrating factor as a function of x, namely, $\mu(x) = x^{-3}$. Multiplication by $\mu(x)$ reduces the given equation to an exact equation $M_1\,\mathrm{d}x + N_1\,\mathrm{d}y = 0$, where

$$M_1(x,y) = y\,e^{xy} - 2x^{-3}y^3 \qquad \text{and} \qquad N_1(x,y) = x\,e^{xy} + 3x^{-2}y^2.$$

Integrating the latter with respect to y, we get

$$\psi(x,y) = \int N_1(x,y)\,\mathrm{d}y = e^{xy} + x^{-2}y^3 + k(x) \quad \Longrightarrow \quad \psi_x = y\,e^{xy} - 2x^{-3}y^3 + k'(x).$$

Since $k'(x) \equiv 0$, we obtain the potential function and the general solution:

$$e^{xy} + x^{-2}y^3 = C.$$

(d) With $M = 4$ and $N = -e^{y-2x}$, we have $M_y = 0$ and $N_x = 2\,e^{y-2x}$. Since the ratio

$$\frac{M_y - N_x}{N} = \frac{2\,e^{y-2x}}{e^{y-2x}} = 2$$

is a function on x (as well as on any other variable because it is a constant), there exists an integrating factor as a function of x: $\mu(x) = e^{2x}$. Upon multiplication by $\mu(x)$, we obtain an exact equation

$$4\,e^{2x}\,\mathrm{d}x - e^y\,\mathrm{d}y = 0.$$

Actually, it is a separable equation, so simple integration yields the general solution:

$$2\,e^{2x} - e^y = C.$$

3.5 (20 pts) Find an integrating factor as a function of y only and determine the general solution for the given differential equations (a and b are constants).

- [2pts] Compute the integrating factors
- [2pts] Integral
- [1pt] Final solution

(a) $(y+3)\,\mathrm{d}x - (x-y)\,\mathrm{d}y = 0$; **(b)** $\left(\frac{y}{x} - 1\right)\,\mathrm{d}x + \left(2y^2 + 1 + \frac{x}{y}\right)\,\mathrm{d}y = 0$;
(c) $(2xy^2 + 3y)\,\mathrm{d}x - 3x\,\mathrm{d}y = 0$; **(d)** $y(x+y+1)\,\mathrm{d}x + x(x+3y+2)\,\mathrm{d}y = 0$.

Solution: In all problems, subscript M_x means the partial derivative $M_y = \frac{\partial M}{\partial y}$ and correspondingly $N_x = \frac{\partial N}{\partial x}$. Integrating factor as a function of y can be obtained explicitly:

$$\mu(y) = \exp\left\{ -\int \frac{M_y - N_x}{M}\,\mathrm{d}y \right\}. \tag{4}$$

(a) Let $M(x,y) = y+1$ and $N(x,y) = y-x$. Since $M_y = 1$ and $N_x = -1$, the given equation is not exact. Since the ratio

$$\frac{M_y - N_x}{N} = \frac{2}{y+3}$$

is a function on y only, there exists an integrating factor

$$\mu(y) = \exp\left\{ -\int \frac{2}{y+3}\,\mathrm{d}y \right\} = (y+3)^{-2}.$$

Multiplying both sides of the given differential equation by $\mu(y)$, we get an exact equation:

$$\frac{\mathrm{d}x}{y+3} + \frac{y-x}{(y+3)^2}\,\mathrm{d}y = 0, \quad \text{with} \quad M = (y+3)^{-1}, \quad N = \frac{y-x}{(y+3)^2}.$$

Integrating $\psi_x = (y+3)^{-1}$ with respect to x, we obtain

$$\psi(x,y) = \frac{x}{y+3} + h(y) \qquad \Longrightarrow \qquad \psi_y = -\frac{x}{(y+3)^2} + h'(y) = N = \frac{y-x}{(y+3)^2}.$$

Therefore, $h'(y) = \frac{y}{(y+3)^2}$. Integration yields $h(y) = 3\,(y+3)^{-1} + \ln(3+y)$. Hence, the general solution becomes

$$\frac{x+3}{y+3} + \ln|3+y| = C.$$

(b) With $M = y/x - 1$ and $N = 2y^2 + 1 + x/y$, we have $M_y = 1/x$ and $N_x = 1/y$. Therefore the ratio

$$\frac{M_y - N_x}{M} = \frac{1/x - 1/y}{y/x - 1} = \frac{1}{y}$$

is a function of y only. So there exists an integrating factor $\mu(y) = y^{-1}$, upon multiplication by it, we get an exact equation:

$$\frac{y-x}{xy}\,\mathrm{d}x + \left(2y + \frac{1}{y} + \frac{x}{y^2}\right)\mathrm{d}y = 0.$$

Therefore there exists a potential function $\psi(x,y)$ such that

$$\psi_x = \frac{y-x}{xy} \qquad \text{and} \qquad \psi_y = 2y + \frac{1}{y} + \frac{x}{y^2}.$$

Integrating the latter with respect to y, we obtain

$$\psi(x,y) = y^2 + \ln|y| - \frac{x}{y} + k(x) \qquad \Longrightarrow \qquad k'(x) = \frac{1}{x}.$$

Hence, the general solution becomes

$$\psi(x,y) \equiv y^2 + \ln|y| + \ln|x| - \frac{x}{y} = C.$$

(c) With $M = 2xy^2 + 3y$ and $N = -3x$, we have $M_y = 4xy + 3$ and $N_x = -3$. Since the ratio

$$\frac{M_y - N_x}{M} = \frac{4xy + 6}{2xy^2 + 3y} = \frac{2\,(2xy+3)}{y\,(2xy+3)} = \frac{2}{y}$$

is a function of y only, we find an integrating factor $\mu(y) = y^{-2}$. Upon its multiplication, we get an exact equation

$$\left(2x + \frac{3}{y}\right) \mathrm{d}x - \frac{3x}{y^2}\,\mathrm{d}y = 0.$$

Therefore, there exists a potential function $\psi(x,y)$ such that its partial derivatives are multiples of differentials:

$$\psi_x = 2x + \frac{3}{y} \qquad \text{and} \qquad \psi_x = -\frac{3x}{y^2}.$$

Integrating the former, we obtain

$$\psi(x,y) = x^2 + \frac{3x}{y} + h(y) \qquad \Longrightarrow \qquad h'(y) = 0.$$

So the general solution becomes

$$x^2 + \frac{3x}{y} = C.$$

(d) With $M = y(x+y+2)$ and $N = x(x+3y+4)$, we have $M_y = x+2y+2$ and $N_x = 2x+3y+4$. Since the ratio

$$\frac{M_y - N_x}{M} = -\frac{2+x+y}{y(x+y+2)} = -\frac{1}{y}$$

is a function of y only, we find an integrating factor $\mu(y) = y$. Upon multiplication by $\mu(y)$, we get an exact equation

$$y^2\,(x+y+2)\,\mathrm{d}x + \left(x^2y + 3xy^2 + 4xy\right)\mathrm{d}y = 0.$$

For a potential function $\psi(x,y)$, we have

$$\psi_x = y^2\,(x+y+2) \qquad \text{and} \qquad \psi_y = x^2y + 3xy^2 + 4xy.$$

Integrating the latter with respect to y, we obtain

$$\psi(x,y) = \frac{x^2y^2}{2} + xy^3 + 2xy^2 + k(x) \qquad \Longrightarrow \qquad k'(x) = 0.$$

Therefore, the general solution becomes

$$\frac{x^2y^2}{2} + xy^3 + 2xy^2 = C.$$