APMA 0330 — Applied Mathematics - I

Brown University **Fall, 2017**

Solutions to Homework, Set 2 **Due September 27**

2.1 (10 pts) A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

Solution: The surface area of a sphere is $A = 4\pi\, r^2$ [2 pts] and its volume is $V = \frac{4}{3}\,\pi\, r^3$ [2 pts] . Expressing volume through the area, we get

$$V = \frac{4}{3}\,\pi\, r^3 = \frac{4}{3}\,\pi \left(\frac{A}{4\pi}\right)^{3/2}.$$

Therefore, the required differential equation becomes [6 pts]

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -k\, V^{2/3}$$

for some constant k.

2.2 (10 pts) Newton's law of cooling states that the temperature $u(t)$ of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings (the ambient air temperature in most cases):

$$\dot{u}(t) = -k\,(u - T)\,,$$

where T is the ambient temperature and k is a positive constant. Suppose that the initial temperature of the object is $u(0) = u_0$, find its temperature at any time t.

Solution: We compute:

$$\begin{aligned} \frac{\mathrm{d}u}{u - T} &= -k\mathrm{d}t \qquad [3\,pts] \\ \ln(u - T) - \ln(u_0 - T) &= -k(t - t_0) \qquad [3\,pts] \text{ for integration} \\ u(t) &= T + (u_0 - T)e^{-k(t - t_0)} \end{aligned}$$

Or, since we have $t_0 = 0$:

$$u(t) = T + (u_0 - T)e^{-kt} \qquad [4\,pts]$$

2.3 (20 pts) Consider a falling object of mass 5 kg that experiences the drag force, which is assumed to be proportional to the square of the velocity (denoted by v):

$$\dot{v} = \left[49^2 - v^2\right]/245.$$

(a) Determine an equilibrium solution.

(b) Plot a slope field for the given differential equation using one of your lovely software package. Provide the codes of your plot or state what resources did you use. Based on the direction field, determine the behavior of $v(t)$ as $t \to \infty$.

(c) Find the limiting velocity $v_\infty = \lim_{t\to\infty} v(t)$ if initially $v(0) = 0$, and determine the time that must elapse for the object to reach 98% of its limiting velocity.

(d) Find the time it takes the object to fall 300 m.

Solution: **(a)** The equilibrium is achieved when

$$\frac{49^2 - v^2}{245} = 0, \quad [1\,\textit{point}]$$

i.e., $v = \pm 49$. However, note that $v = -49$ is nonphysical: we designate positive velocity as downwards velocity, so $v = -49$ would be falling upwards. Thus, the only equilibrium in which we are interested is $v = 49$ [2 pts] . **Reduction:** 1 point off if ± 49.

(b) We have that $v \to 49$ as $t \to \infty$ for all physical trajectories. Note that with this model, if the object were moving upwards fast enough, it would continue to accelerate upwards forever!

Now we plot using *Mathematica:*

```
VectorPlot[{1, (49^2 - y^2)/245}, {x, -100, 100}, {y, -100, 100},
 VectorPoints -> 20, VectorStyle -> Arrowheads[0.028]]
```

Then we do the same job using MATLAB:

```
[t, v] = meshgrid(0:.5:10,40:1:60);
dv = (49^2-v.^2)/245;
dt = ones(size(dy));
dvu = dv./sqrt(dt.^2 + dv.^2);
dtu = dt./sqrt(dt.^2 + dv.^2);
quiver(t,v,dtu,dvu,1);
```

(c) If $v(0) = 0$, then v_∞ is indeed 49. To find the elapsed time we first solve the differential equation

$$\begin{aligned} \frac{\mathrm{d}v}{49^2 - v^2} &= \frac{\mathrm{d}t}{245} \\ \frac{\mathrm{d}v}{98}\left(\frac{1}{49+v} + \frac{1}{49-v}\right) &= \frac{\mathrm{d}t}{245} \quad [2\,\textit{pts}] \text{ for separation of variables.} \end{aligned}$$

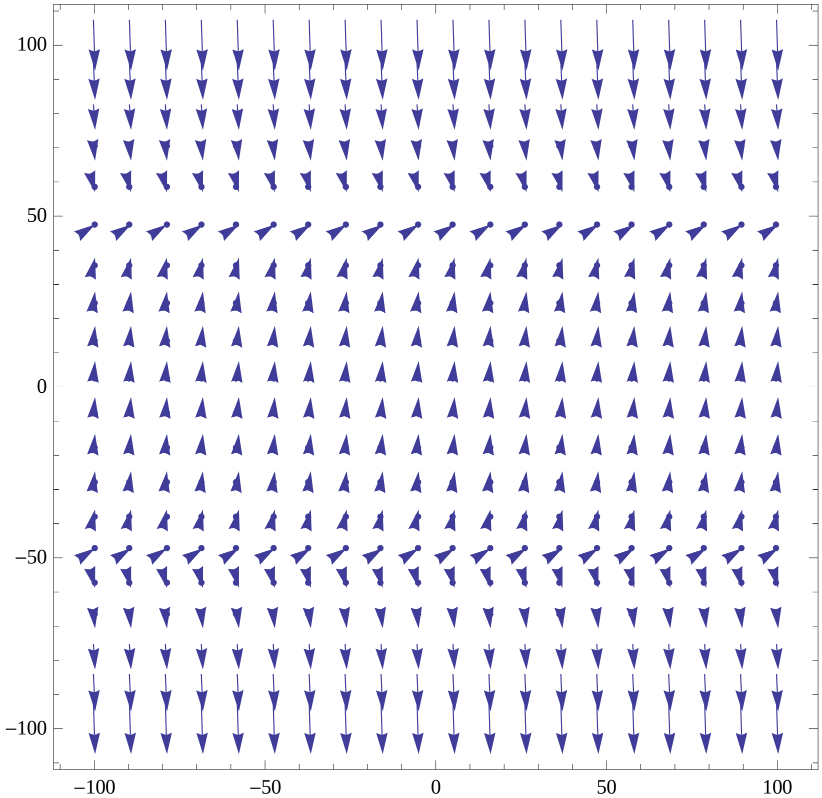

Figure 1: Direction field for Problem **2.3**, plotted with *Mathematica*.

Then we integrate

$$\begin{aligned}
\int \left(\frac{1}{49+v} + \frac{1}{49-v}\right) \mathrm{d}v &= \int \frac{2}{5}\,\mathrm{d}t \\
\frac{1}{98}\left[\ln(49+v) - \ln(49-v)\right] &= \frac{t}{245} \\
\ln\left(\frac{49+v}{49-v}\right) &= \frac{2t}{5} \\
\frac{49+v}{49-v} &= e^{\frac{2}{5}t} \\
49+v &= (49-v)e^{\frac{2}{5}t} \quad \text{[3 } pts\text{] for intermediate steps} \\
v(1+e^{\frac{2}{5}t}) &= 49(1-e^{\frac{2}{5}t}) \\
v &= 49\,\frac{e^{2t/5-1}}{1+e^{2t/5}} \\
&= 49\tanh\left(\frac{t}{5}\right) \quad \text{[2 } pts\text{] for solution}
\end{aligned}$$

We seek t such that $v(t) = 0.98 \cdot v_\infty = 0.98 \cdot 49$, i.e.,

$$\begin{aligned} 0.98 \cdot 49 &= 49 \tanh\left(\frac{t}{5}\right) \\ t &= 5 \tanh^{-1}(0.98) \\ &\approx 11.5 \quad [2\,pts] \end{aligned}$$

(d) We can arbitrarily define the starting point as $x_0 = x(0) = 0$. then the position of the object is the same as the distance traveled, and is given by [2 pts]

$$x(t) = \int_0^t v(s)\mathrm{d}s = \int_0^t 49 \tanh\left(\frac{s}{5}\right) \mathrm{d}s = 245 \ln \cosh\left(\frac{t}{5}\right).$$

We seek t such that $x(t) = 300$, then [2 pts] for answer and [2 pts] for simplification

$$300 = 245 \ln \cosh\left(\frac{t}{5}\right) \implies t = 5 \cosh^{-1}\left(e^{300/245}\right) \approx 9.48.$$

2.4 (10 pts) At a given level of effort, it is reasonable to assume that the rate at which fish are caught depends on the population $P(t)$: the more fish there are, the easier it is to catch them. Thus we assume that the rate at which fish are caught is given by $E\,P(t)$, where E is a positive constant, with units of 1/time, that measures the total effort made to harvest the given species of fish. To include this effect, the logistic equation is replaced by

$$\mathrm{d}P/\mathrm{d}t = r\,(1 - P/K)\,P - EP, \qquad (i)$$

where r and K are positive constants. This equation is known as the **Schaefer model**.

(a) Show that if $E < r$, the there are two equilibrium points $P_1 = 0$ and $P_2 = K\,(1 - E/r) > 0$.

(b) Show that $P = P_1$ is unstable and $P = P_2$ is asymptotically stable. As a confirmation, you may want to draw a direction field for some numerical values of constants r, K, and E.

(c) A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort E and the asymptotically stable population P_2. Find Y as a function of the effort E: the graph of this function is known as the yield–effort curve.

(d) Determine E so as to maximize Y and thereby find the **maximum sustainable yield** Y_m.

Solution:

(a) Equating the slope function to zero, we get [1 point]

$$r\,(1 - P/K)\,P - EP = 0 \implies P\left(r - E - \frac{r}{K}\,P\right) = 0.$$

Therefore, there are two critical points: [1 point]

$$P_1 = 0 \qquad \text{and} \qquad P_2 = K\left(1 - E\frac{1}{r}\right).$$

We observe that for [1 point]

$$E < r \implies -\frac{E}{r} > -1 \implies P_2 > 0.$$

(b) We define $f(P) = r\,(1 - P/K)\,P - EP$, then we compute its derivative [1 point]

$$\frac{\mathrm{d}f(P)}{\mathrm{d}P} = \frac{\mathrm{d}}{\mathrm{d}P}(r\,(1 - P/K)\,P - EP) = r\,(1 - P/K) - rP/K - E.$$

Evaluating at P_1 and P_2 we obtain

$$\frac{\mathrm{d}f(P_1)}{\mathrm{d}P} = r\left(1 - P_1\frac{1}{K}\right) - \frac{r}{K}P_1 - E = r - E > 0. \quad [1\,point]$$

$$\begin{aligned}\frac{\mathrm{d}f(P_2)}{\mathrm{d}P} &= r\left(1 - P_2\frac{1}{K}\right) - \frac{r}{K}P_2 - E \quad [1\,point]\\ &= r\left(1 - K\left(1 - \frac{E}{r}\right)\frac{1}{K}\right) - \frac{r}{K}K\left(1 - \frac{E}{r}\right) - E\\ &= r\left(\frac{E}{r}\right) - r\left(1 - \frac{E}{r}\right) - E\\ &= E - r < 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}f(P_1)}{\mathrm{d}P} > 0 &\implies P_1 \text{ is unstable}\\ \frac{\mathrm{d}f(P_2)}{\mathrm{d}P} < 0 &\implies P_2 \text{ is asymptotically stable.}\end{aligned}$$

(c) By definition we have [1 point]

$$Y = E\,P_2 = EK\left(1 - \frac{E}{r}\right).$$

(d) We maximize Y. Differentiating the previous equation and equaling to zero we have [1 point each equation]

$$0 = \frac{\mathrm{d}Y(E)}{\mathrm{d}E} = K\left(1 - \frac{E}{r}\right) - \frac{1}{r}EK \implies E = \frac{r}{2}.$$

Then, [1 point]

$$Y_m = Y\left(\frac{r}{2}\right) = \frac{r}{2}K\left(1 - \frac{1}{2}\right) = \frac{r}{4}K.$$

2.5 (10 pts) Assuming that fish are caught at a constant rate h, its population is modeled by

$$\mathrm{d}P/\mathrm{d}t = r\,(1 - P/K)\,P - h. \qquad (ii)$$

(a) If $h < rK/4$, show that Eq. (ii) has two equilibrium points q_1 and q_2 with $q_1 < q_2$; determine these points.

(b) Show that q_1 is unstable and q_2 is asymptotically stable.

(c) From a plot of the rate function $r\,(1 - P/K)\,P - h$ versus P, show that if the initial population $P(0) > q_1$, then $P(t) \mapsto q_2$ as $t \to \infty$, but that if $P(0) < q_1$, then $P(t)$ decreases as t increases. Note that $P \equiv 0$ is not an equilibrium point, so if $P(0) < q_1$, then extinction will be reached in a finite time.

(d) If $h > rK/4$, show that $P(t)$ decreases to zero as t increases, regardless of the value of $P(0)$.

(e) If $h = rK/4$, show that there is a single equilibrium point $P \equiv K/2$ and that this point is semistable. Thus the maximum sustainable yield is $h_m = rK/4$, corresponding to the equilibrium value $P \equiv K/2$. Observe that h_m has the same value as Y_m.

Solution:

(a) Equating the slope function to zero, we get [1 point]

$$r\,(1 - P/K)\,P - h = 0 \quad \Longrightarrow \quad -P^2\frac{r}{K} + r\,P - h = 0.$$

This is a quadratic equation that has two roots [1 point]

$$q_1 = \frac{r - \sqrt{r^2 - 4\,\frac{r}{K}\,h}}{2\frac{r}{K}} \qquad \text{and} \qquad q_2 = \frac{r + \sqrt{r^2 - 4\,\frac{r}{K}\,h}}{2\frac{r}{K}}$$

when its discriminant is not negative, that is, $r^2 > 4h\,\frac{r}{K}$.

(b) Define $f(P) = r\,(1 - P/K)\,P - h$, then we differentiate [1 point]

$$\frac{\mathrm{d}f(P)}{\mathrm{d}P} = r\,(1 - P/K) - r\,P\,\frac{1}{K} = r - 2r\,P\,\frac{1}{K}.$$

Evaluating at the equilibrium points, we obtain [1 point]

$$\frac{\mathrm{d}f(q_1)}{\mathrm{d}P} = \sqrt{r^2 - 4\,\frac{r}{K}\,h} > 0$$
$$\frac{\mathrm{d}f(q_2)}{\mathrm{d}P} = -\sqrt{r^2 - 4\,\frac{r}{K}\,h} < 0.$$

We conclude that q_1 is unstable and q_2 is asymptotically stable. [1 point]

(c) Plot [1 point]

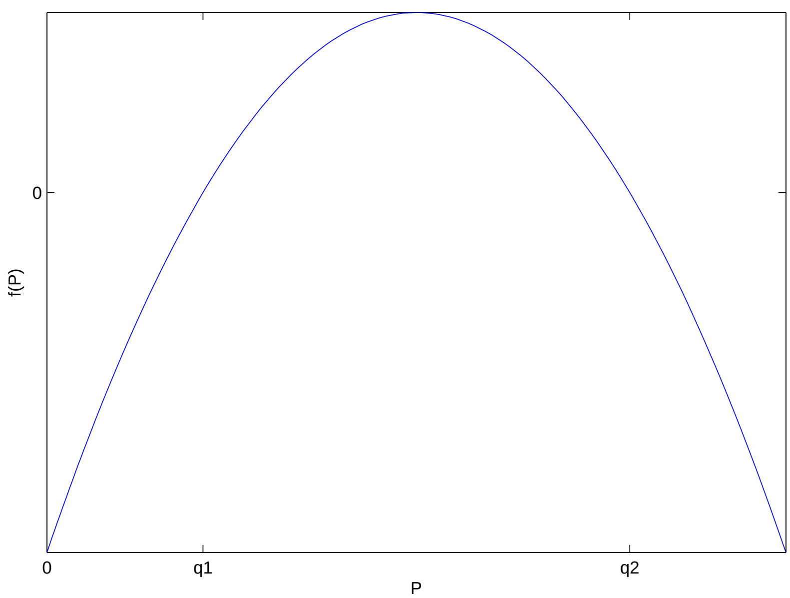

Figure 1: Plot of P vs $f(P) = r\,(1 - P/K)\,P - h$.

From the plot we observe [1 point]

- If $P(0) > q_1$ and $P(0) <= q_2$, we observe in Figure 1 that the values of the function are positive (the slopes are positives), then the solution is increasing towards q_2.
- If $P(0) > q_2$, we observe in Figure 1 that the values of the function are negative, then the solution is decreasing towards q_2 .
- If $0 < P(0) < q_1$, we observe in Figure 1 that the values of the function are negative, then the solution is decreasing, then it will reach 0 in a finite time.

(d) [2 pts] If $h > r\frac{K}{4}$, then the discriminant of the equilibrium points is negative. Therefore there are no critical points and equivalent curve to Figure 1 will be below 0 (meaning that $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$). We can conclude that the solutions are decreasing as t increases independently of the value of $P(0)$.

(e) [2 pts] If $h = r\frac{K}{4}$, then the discriminant is 0 ($q_1 = q_2$) and we have one equilibrium point $q_1 = \frac{K}{2}$. Evaluating the derivative of f in this point, we have

$$\frac{\mathrm{d}f(q_1)}{\mathrm{d}P} = 0.$$

Then we conclude that this point is semistable. We can also see this from the plot of the quadratic function $f(P)$ with one root, since the function is negative for all values of $P \neq q_1$.

2.6 (10 pts) Solve the given differential equation of the form $x\,y' = y\,F(xy)$ by using transformation $v = xy$.

(a) $x\,y' = e^{xy} - y$; (b) $x\,y' = y/(xy+1)$.

Solution: **(a)** [5 pts] Setting $v = xy$, we differentiate and separate variables: [1 point for change of variable]

$$v' = y + xy' = y + e^{xy} - y = e^{xy} = e^{v} \qquad \Longrightarrow \qquad e^{-v}\mathrm{d}v = \mathrm{d}x.$$

Integration yields [1 point for integration and 2 points for integration]

$$-e^{-v} = x + C \qquad \text{or} \qquad x + e^{-xy} = C,$$

where C is a constant of integration. [1 point for final answer]

(b) [5 pts] Differentiation of $v = xy$ yields

$$v' = y + xy' = y + \frac{y}{v+1} = y\left(1 + \frac{1}{v+1}\right) = \frac{v}{x}\,\frac{v+2}{v+1}.$$

Separation of variables yields

$$\frac{v+1}{v(v+2)}\,\mathrm{d}v = \frac{\mathrm{d}x}{x}$$

Integrating the latter, we have

$$\frac{1}{2}\,\ln|v| + \frac{1}{2}\,\ln|v+2| = \ln Cx \qquad \Longleftrightarrow \qquad \ln|v(v+2)| = \ln Cx^2.$$

Exponentiation brings the general solution:

$$v(v+2) = Cx^2 \qquad \text{or} \qquad y(xy+2) = Cx.$$

2.7 **(10 pts)** Solve the differential equation $y' = (4x+y-5)^2$ by using appropriate transformation.

Solution: Setting $v = 4x + y - 5$, we get [3 pts]

$$v' = 4 + y' = 4 + v^2.$$

Separation of variables yields [4 pts]

$$\int \frac{\mathrm{d}v}{4+v^2} = x + C \qquad \Longrightarrow \qquad \frac{1}{2}\,\arctan\frac{v}{2} = x + C.$$

When we return to the original variables, we obtain the general solution: [3 pts]

$$4x + y - 5 = 2\,\tan\left(2x + C\right).$$

2.8 (20 pts) Solve the given differential equation with a homogeneous right-hand side function. Then determine an arbitrary constant that satisfies the auxiliary condition.

(a) $xy\,\mathrm{d}x + (x^2 + 3y^2)\,\mathrm{d}y = 0, \quad y(1) = 1;$

(b) $(y + \sqrt{x^2 + y^2})\,\mathrm{d}x - 2x\,\mathrm{d}y = 0, \quad y(1) = 0;$

(c) $(x - y)\,\mathrm{d}x + (3x + 2y)\,\mathrm{d}y = 0, \quad y(2) = 1;$

(d) $(y^2 + 3xy)\,\mathrm{d}x - 2x^2\,\mathrm{d}y = 0, \quad y(1) = 1.$

Solution: In all problems, we use substitution $y = x\,v$, where $v = v(x)$ is unknown function to be determined. Its differential becomes $\mathrm{d}y = v\,\mathrm{d}x + x\,\mathrm{d}v$.

(a) [5 pts] Using substitution $y = x\,v$, we get

$$v\,\mathrm{d}x + \left(1 + 3v^2\right)[v\,\mathrm{d}x + x\,\mathrm{d}v] = 0.$$

Therefore, separation of variables yields

$$-\frac{\mathrm{d}x}{x} = \frac{(1 + 3v^2)}{v\,(2 + 3v^2)}\,\mathrm{d}v,$$

which upon integration gives

$$-\ln Cx = \frac{1}{4}\left(2\,\ln v + \ln(2 + 3v^2)\right) = \frac{1}{6}\,\ln(3 + 4v^2)\,v^4.$$

Raising to exponent, we get

$$\frac{C}{x^4} = (2 + 3v^2)\,v^2 = \left(2 + \frac{3y^2}{x^2}\right)\frac{y^2}{x^2}.$$

Multiplication by x^4 yields the general solution:

$$C = \left(2x^2 + 3y^2\right)y^2.$$

To satisfy the initial condition $y(1) = 1$, we set $C = 5$.

(b) [5 pts] Upon substitution $y = x\,v$, we get

$$\left(v + \sqrt{1 + v^2}\right)\mathrm{d}x = 2\,[v\,\mathrm{d}x + x\,\mathrm{d}v]\,.$$

Separation of variables and integration yields

$$\frac{2}{\sqrt{1 + v^2} - v}\,\mathrm{d}v = \frac{\mathrm{d}x}{x} \qquad \Longrightarrow \qquad v\left(v + \sqrt{1 + v^2}\right) + \operatorname{arcsinh} v = \ln Cx.$$

With old variables $v = y/x$, we obtain the general solution

$$\ln Cx = \frac{y}{x}\left(\frac{y}{x} + \frac{\sqrt{x^2 + y^2}}{x}\right) + \operatorname{arcsinh}\left(\frac{y}{x}\right).$$

Using the initial condition $y(1) = 0$, we derive $C = 1$ and the particular solution becomes

$$x^2 \ln x = y\left(y + \sqrt{x^2 + y^2}\right) + x^2 \operatorname{arcsinh}\left(\frac{y}{x}\right).$$

(c) [5 pts] Upon substitution $y = x\,v$, we get

$$v + x\,v' = \frac{v-1}{3+2v} \qquad \Longrightarrow \qquad x\,v' = \frac{v-1}{3+2v} - v = -\frac{1+2v+2v^2}{3+2v}.$$

Separation of variables and integration yields

$$\frac{3+2v}{(2v^2+2v+1}\,\mathrm{d}v = -\frac{\mathrm{d}x}{x} \qquad \Longrightarrow \qquad 4\,\arctan(1+2v) + \ln\left(1+2v+2v^2\right) = -2\,\ln x + C.$$

Uniting two logarithms, we get

$$C = 4\,\arctan(1+2v) + \ln x^2\left(1+2v+2v^2\right)$$

From the initial condition, it follows that $v(2) = 1/2$ and we determine the value of arbitrary constant C to be $C = 4\,\arctan 2 + \ln 10$. Since $v = y/x$, we obtain the general solution in implicit form:

$$4\,\arctan 2 + \ln 10 = 4\,\arctan\frac{2y+x}{x} + \ln\left(x^2+2xy+2y^2\right).$$

(d) [5 pts] Upon substitution $y = x\,v$, we get

$$\left(v^2+3v\right)\mathrm{d}x = 2\left[x\,\mathrm{d}v + v\,\mathrm{d}x\right] \qquad \Longrightarrow \qquad \left(v^2+v\right)\mathrm{d}x = 2x\,\mathrm{d}v.$$

Separation of variables and integration yields

$$\frac{\mathrm{d}v}{v^2+v} = \frac{\mathrm{d}x}{2x} \qquad \Longrightarrow \qquad \ln(v) - \ln(1+v) = \ln\frac{v}{1+v} = \ln Cx.$$

Raising to exponent, we have

$$\frac{v}{1+v} = C\,x^{1/2}.$$

Since $v = y/x$, we obtain the general solution:

$$\frac{y}{x+y} = C\,x^{1/2}.$$

From the initial condition follows that $C = 1/2$.

Grade distribution for the last problem:
[2 pts] for change of variable
[2 pts] for separation of variables
[2 pts] for integration
[2 pts] for correct use of initial condition
[2 pts] for final answer