1 Problem 1.1 (20 pts):

Consider a population of field mice who inhabit a certain rural area. Suppose that several owls live in the same neighborhood and they kill 15 mice per day. Then the mouse population $p(t)$, t is measured in months, satisfies the differential equation

$$\dot{p} \equiv \mathrm{d}p/\mathrm{d}t = p/3 - 360.$$

1. Plot a direction field for the given differential equation using one of your lovely software package. Provide the codes of your plot or state what resources did you use. Based on the slope field, determine the behavior of $p(t)$ as $t \to \infty$.
2. [3 pts] Determine an equilibrium solution.
3. Find the time at which the population becomes extinct if $p(0) = 800$.
4. Find the initial population $p(0)$ if the population is to become extinct in 2 years.

Solution [3 pts] for coding;
[2 pts] for graph (suitable range, equilibrium solution).
Using Mathematica,

```
StreamPlot[{1, y/3 - 360}, {t, 0, 10}, {y, 1070, 1090},
 StreamPoints -> Fine]
```

or MATLAB

```
close all
[t,p] = meshgrid(-10:1:10,1000:10:1100); %create mesh grid
dpdt = p./3 - 360; %ode
new_t = ones(size(t)); %create array of ones

figure
quiver(t,p,new_t,dpdt) %plot

[t2,v] = meshgrid(-75:10:75,-75:10:75);
dvdt = (49^2-v.^2)/245;
new_t2 = ones(size(t2));

figure
quiver(t2,v,new_t2,dvdt)
```

we plot the slope field. The solution tends to $+\infty$, if it starts above 1080, or $-\infty$, if it starts below 1080.

The equilibrium solution is $p^* = 1080$. Since the given equation is autonomous, we separate variables [4 pts]

$$\frac{\mathrm{d}p}{p/3 - 360} = \mathrm{d}t,$$

which upon integration yields [3 pts]

$$p(t) = 1080 + (p_0 - 1080)\, e^{(t-t_0)/3},$$

where the initial condition $p(t_0) = p_0$ has been incorporated into the solution. Using the initial value $p(0) = 900$, we get

$$p(t) = 1080 - 180\, e^{t/3}.$$

If we need to find time t for which $p(t) = 0$, we have to solve the equation $p(t) = 0$ with respect to t. This leads to

$$0 = 1080 - 180\, e^{t/3} \qquad \Longrightarrow \qquad t = 3\, \ln 6 \approx 5.37528.$$

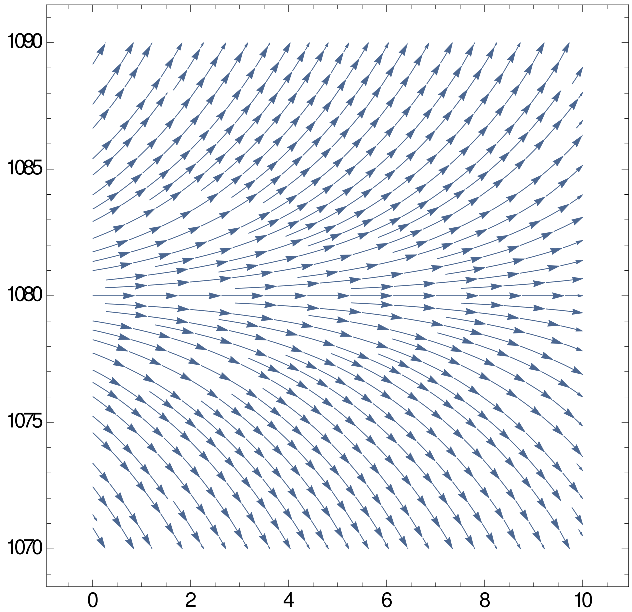


Figure 1: Direction field for problem 1, plotted with Mathematica.

To answer the last question, we have to solve the equation $p(24) = 0$. Using the explicit solution, we find

$$\begin{aligned} p(24) &= 1080 + (p_0 - 1080)\, e^{24/3} \quad [2\,pts] \\ 0 &= 1080 + (p_0 - 1080)\, e^{8}, \quad [2\,pts] \\ p_0 &= 1080\left(1 - e^{-8}\right) \approx 1079.64 \quad [1\,pts] \end{aligned}$$

2 Problem 1.2 (10 pts):

Suppose that the population of fish is modeled by the logistic equation:

$$\dot{P}(t) = 0.1\, P\,(1 - P/500)\,.$$

1. Find critical points and identify them as unstable of asymptotically stable. [5 pts]
2. Find the solution of the above logistic equation subject to the initial condition $P(0) = 250$. [5 pts]

Solution: The general solution of the given logistic equation is

$$P(t) = \frac{500\, e^{t/10}}{C + e^{t/10}},$$

where C is an arbitrary constant. Therefore, the solution to the given initial value problem becomes

$$P(t) = \frac{500\, e^{t/10}}{1 + e^{t/10}}.$$

The logistic equation has two critical points: $P = 0$ (unstable) and $P = 500$, which is asymptotically stable.

3 Problem 1.3 (30 pts):

Consider the initial value problem

$$y' = x - y, \qquad y(0) = 1.$$

Algorithm 1 Exercise 1.1 (MATLAB)

```
% configuration arguments
t0 = 0; t1 = 12; ts = 1;
p0 = 500; p1 = 1300; ps = 50;
% generate a grid of points where the slop element will be drawn
[ t, p ] = meshgrid( t0:ts:t1, p0:ps:p1 );
dp = p /3 - 360;
hold on;
% MATLAB messes up with the arrow head if the aspect ratio of the axes
% differs substantially from 1, a simple solution is to turn it off
figure = quiver( t, p, ones(size(dp)), dp, 0.8, 'ShowArrowHead', 'off' );
% add title, adjust axis, etc...
title('direction field for exercise 1.3');
axis( [ t0, t1, p0, p1 ] );
xlabel('t / month', 'FontSize', 16 );
ylabel('population', 'FontSize', 16 );
hold off;
```

1. Convert the given initial value problem to the integral equation:

$$y(x) = y_0 + \int_{x_0}^{x} f\left(s, y(s)\right) \mathrm{d}s,$$

by identifying x_0, y_0, and the slope function $f(x, y)$. [10 pts]

2. Use Picard's iterations

$$\phi_{n+1}(x) = y_0 + \int_0^x f\left(s, \phi_n(s)\right) \mathrm{d}s$$

to find the unique solution of the given initial value problem. [20 pts]

Solution

1. We convert the given initial value problem to the integral equation:

$$y(x) = 1 + \int_0^x \left(s - y(s)\right) \mathrm{d}s,$$

where $x_0 = 0$, $y_0 = 1$, and the slope function $f(x, y) = x - y$.

2. We use Picard's iterations [10 pts]

$$\phi_{n+1}(x) = 1 + \int_0^x \left(s - \phi_n(s)\right) \mathrm{d}s, \qquad \phi_0 = 1.$$

Calculations show that

$$\begin{aligned}
\phi_1(x) &= 1 + \int_0^x (s-1)\,\mathrm{d}s = 1 - x + \frac{x^2}{2},\\
\phi_2(x) &= 1 + \int_0^x (s - \phi_1(s))\,\mathrm{d}s = 1 - x + x^2 - \frac{x^3}{6} = 2\left(1 - x + \frac{x^2}{2}\right) - 1 + x - \frac{x^3}{6},\\
\phi_3(x) &= 1 + \int_0^x (s - \phi_2(s))\,\mathrm{d}s = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24},\\
\phi_4(x) &= 1 + \int_0^x (s - \phi_3(s))\,\mathrm{d}s = 1 - x + x^2 - 2\,\frac{x^3}{3!} + 2\,\frac{x^4}{4!} - \frac{x^5}{5!},\\
\phi_5(x) &= 1 + \int_0^x (s - \phi_4(s))\,\mathrm{d}s = 1 - x + x^2 - 2\,\frac{x^3}{3!} + 2\,\frac{x^4}{4!} - 2\,\frac{x^5}{5!} + \frac{x^6}{6!},\\
\phi_6(x) &= 1 + \int_0^x (s - \phi_5(s))\,\mathrm{d}s = 1 - x + x^2 - 2\,\frac{x^3}{3!} + 2\,\frac{x^4}{4!} - 2\,\frac{x^5}{5!} + 2\,\frac{x^6}{6!} - \frac{x^7}{7!},
\end{aligned}$$

and so on. So we see that the general term $\phi_n(x)$ contains the truncated series for $2\,e^{-x} = 2\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)$. Therefore, we conclude that the solution becomes [10 pts]

$$y(x) = 2\,e^{-x} - 1 + x.$$

Algorithm 2 Exercise 1.3 (*Mathematica*)

```
y1[x_] = 1 - x + x^2 - x^3/3 + 2*x^4 /4! - 2*x^5 /5! + 2*x^6 /6! -
  2*x^7 /7!
y0[x_] = 2*Exp[-x] - 1 + x
Plot[{y1[x], y0[x]}, {x, 0, 5},
 PlotStyle -> {{Thick, Blue}, {Thick, Orange}}]
```

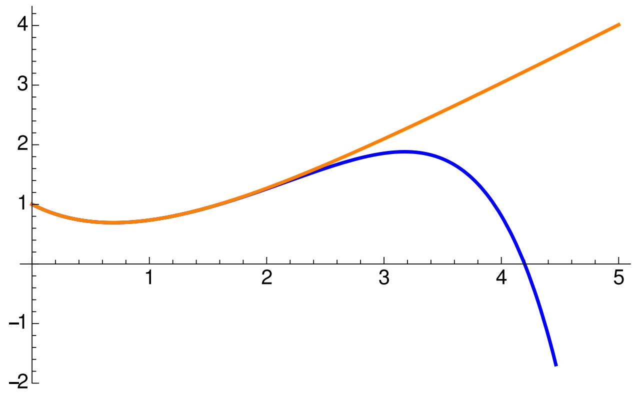


Figure 2: Exact solution and Picard approximation with 8 terms, plotted with Mathematica.

4 Problem 1.4 (16 pts):

A certain drug is being administrated intravenously to a hospital patient. Fluid containing 6 mg/cm^3 of the drug enters the patient's bloodstream at a rate of 120 cm^3/h. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate of 0.5 (h)$^{-1}$.

(a) Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug that is present in the bloodstream at any time.

(b) How much of the drug is present in the bloodstream after a long time?

Grading:
[4 pts] for explanation;
[4 pts] equation
[4 pts] graph
[4 pts] code

Solution:

(a) Let $m(t)$ mg be the total amount of drug in bloodstream at time t. Consider a short time interval $[t, t + \Delta t]$. The increment of the amount of the drug that is present in the bloodstream can be calculated in two ways:

$$m(t + \Delta t) - m(t)$$

and

$$6 \times 120\Delta t - 0.5m\Delta t$$

Therefore, we have

$$m(t + \Delta t) - m(t) = 6 \times 120\Delta t - 0.5m\Delta t.$$

Dividing both sides by Δt and taking the limit: $\Delta t \to 0$, we get

$$m'(t) = 720 - 0.5m(t)$$

(b) First, we plot the direction field using *Mathematica:*

```
StreamPlot[{1, 720 - 0.5*m}, {t, 10, 20}, {m, 1430, 1450}]
```

From the direction field, we know that after a long time, the drug present in bloodstream should be stable, and there should be $m'(t) = 0$. Then we have

$$720 - 0.5m(t) = 0 \qquad \Longrightarrow \qquad m(t) = 1440.$$

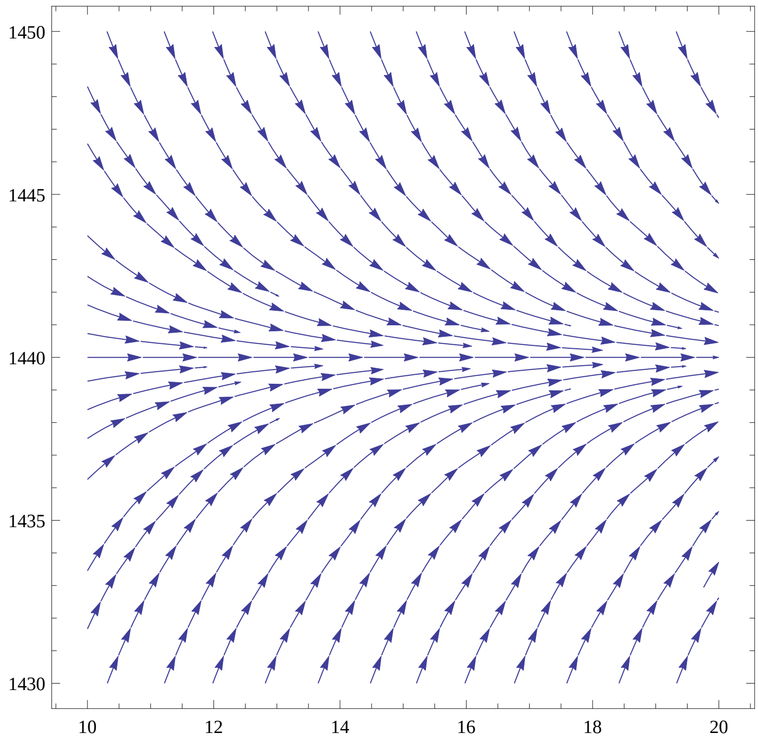


Figure 3: Direction field for Problem 1.4, plotted with *Mathematica*.

5 Problem 1.5 (24 pts):

Consider an electrical circuit containing a capacitor, resistor, and battery. The charge $Q(t)$ on the capacitor satisfies the equation

$$R\,\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = V,$$

where R is the (constant) resistance, C is the (constant) capacitance, and V is the constant voltage supplied by the battery.

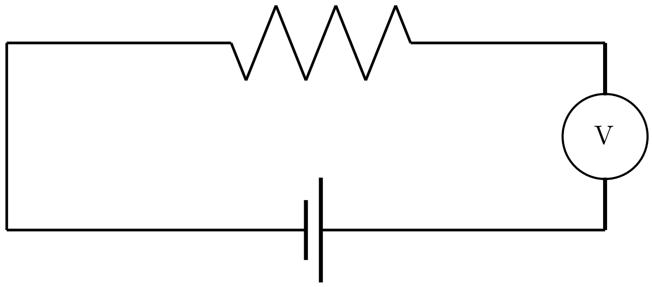


Figure 4: The electric circuit of Problem 1.5.

(a) If $Q(0) = 0$, find $Q(t)$ at any time, and sketch the graph of Q versus t.

(b) Find the limiting value Q_L that $Q(t)$ approaches after a long time.

(c) Suppose that $Q(t_1) = Q_L$ and that at time $t = t_1$ the battery is removed and the circuit is closed again. Find $Q(t)$ for $t > t_1$ and sketch its graph.

Solution: **(a)** The corresponding initial value problem is:

$$\begin{cases} R\,\dfrac{\mathrm{d}Q}{\mathrm{d}t} + \dfrac{Q}{C} = V \\ \\ Q(0) = 0 \end{cases}$$

We subtract $\frac{Q}{C}$ from both sides:

$$R\,\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{Q}{C} + V$$

The equation is clearly separable. Gathering the Q terms on the left-hand side and the t terms on the right-hand side, we get

$$R\,\frac{\mathrm{d}Q}{-\frac{Q}{C} + V} = \mathrm{d}t$$

Now integrate both sides:

$$\int R\frac{\mathrm{d}Q}{-\frac{Q}{C} + V} = \int \mathrm{d}t$$

The right-hand side is a simple integral, the left-hand side is a little trickier but still doable:

$$-RC\ln\left(-\frac{Q}{C} + V\right) = t + k_1$$

In the above question, k is the constant of integration. Moving the $-RC$ to the right-hand side: [4 pts]

$$\ln\left(-\frac{Q}{C} + V\right) = -\frac{t + k_1}{RC}.$$

Exponentiate both sides:

$$-\frac{Q}{C} + V = e^{-\frac{t+k_1}{RC}} = e^{-\frac{t}{RC}}\,e^{-\frac{k_1}{RC}} = k_2 e^{-\frac{t}{RC}}.$$

Here, $k_2 = e^{-\frac{k_1}{RC}}$ is a new constant, for convenience. Now, solving for Q: [2 pts]

$$Q = -C\left(k_2 e^{-\frac{t}{RC}} - V\right)$$

Now we apply the initial condition $Q(0) = 0$ to solve for the constant k_2: [1 pt] for $C = 0$

$$0 = -C\left(k_2 e^{-\frac{0}{RC}} - V\right) = -C\left(k_2 - V\right) \implies C = 0 \text{ or } k_2 - V = 0.$$

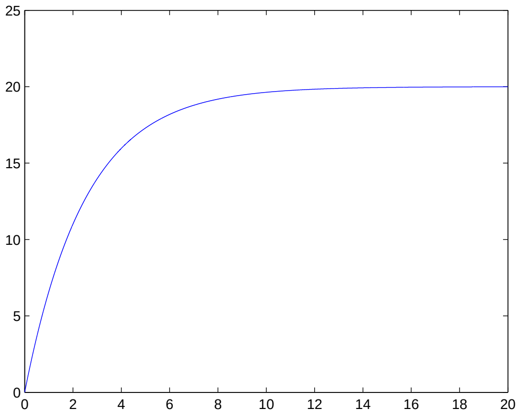
Presumably our capacitance C is nonzero, we thus have that $k_2 = V$. So the solution to our initial value problem is:

$$Q(t) = -C\left(Ve^{-\frac{t}{RC}} - V\right)$$

Cleaning it up a bit:

$$\boxed{Q(t) = CV\left(1 - e^{-\frac{t}{RC}}\right) \qquad t \geq 0}$$

[2 pts] for graph; hand sketch is ok.



In order to sketch a graph of Q versus t, we need to pick some actual values for C, V, and R. The specific values don't matter too much here; we just want to get an idea of what the graph looks at. Arbitrarily choosing $C = 2.5, V = 8, R = 1$, we obtain: The above graph was generated using the following MATLAB code:

```
t = linspace(0,20,10000);
C = 2.5;
V = 8;
R = 1;
Q = C*V*(1-exp(-t/(R*C)));
plot(t,Q)
axis([0, 20, 0, 25])
```

(b)

$$Q_L = \lim_{t\to\infty}\left(CV\left(1 - e^{-\frac{t}{RC}}\right)\right) = \lim_{t\to\infty}\left(CV - CVe^{-\frac{t}{RC}}\right)$$
$$= \lim_{t\to\infty}(CV) - \lim_{t\to\infty}\left(CVe^{-\frac{t}{RC}}\right) = CV - 0 = CV \quad [3\,pts]$$

So, since the exponential term goes to zero as t goes to infinity, we find that $\boxed{Q_L = CV}$.

(c) We now have a different initial value problem. Since the battery has been removed, the voltage $V = 0$. So our initial value problem is:
[2 pts] for separation of variables
[2 pts] for integration

$$\begin{cases} R\dfrac{\mathrm{d}Q}{\mathrm{d}t} + \dfrac{Q}{C} = 0 \\ \\ Q(t_1) = Q_L = CV \end{cases}$$

Our differential equation is again separable, and is relatively straightforward:

$$R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = 0 \implies R\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{Q}{C} \implies \frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{Q}{RC}$$
$$\implies \frac{\mathrm{d}Q}{Q} = -\frac{1}{RC}\mathrm{d}t \implies \int \frac{1}{Q}\mathrm{d}Q = \int -\frac{1}{RC}\mathrm{d}t \implies \ln(Q) = -\frac{t}{RC} + k_1$$
$$\implies Q = e^{-\frac{t}{RC}+k} = e^{-\frac{t}{RC}}e^{k_1} = k_2 e^{-\frac{t}{RC}}$$

We now apply the initial condition to solve for the constant of integration k_2:

$$Q_L = CV = k_2 e^{-\frac{t_1}{RC}} \implies k_2 = CVe^{\frac{t_1}{RC}}$$

Cleaning things up

$$Q(t) = CVe^{\frac{t_1}{RC}}e^{-\frac{t}{RC}} = CVe^{\frac{t_1-t}{RC}} = CVe^{\frac{-(t-t_1)}{RC}}$$

So the solution to our initial value problem is $C = 2.5, V = 8, R = 1, t_1 = 10$:

$$\boxed{Q(t) = CVe^{\frac{-(t-t_1)}{RC}}, \qquad t \geq t_1} \quad [2\,pts]$$

To plot, we again arbitrarily pick values. Using $C = 2.5, V = 8, R = 1, t_1 = 10$: The above graph was generated using the following MATLAB code:

```
t = linspace(0,20,10000);
C = 2.5;
V = 8;
R = 1;
t1 = 10;
Q = C*V*exp(-(t-t1)/(R*C));
plot(t,Q)
axis([0, 20, 0, 25])
```

[2 pts] for graph.
Using *Mathematica:*

```
V[t_] = Piecewise[{{8, 0 < t < 10}}]
s = DSolve[{q'[t] + q[t]/2.5 == V[t], q[0] == 0}, q[t], t]
Plot[Evaluate[q[t] /. s], {t, 0, 25}, PlotStyle -> {Blue, Thick}]
```