NAME: ____

APMA 0330 — Applied Mathematics - I

Brown University | **Fall, 2017**
Homework, Set 2 | **Due September 27**

2.1 **(10 pts)** A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

2.2 **(10 pts)** Newton's law of cooling states that the temperature $u(t)$ of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings (the ambient air temperature in most cases):

$$\dot{u}(t) = -k\,(u - T)\,,$$

where T is the ambient temperature and k is a positive constant. Suppose that the initial temperature of the object is $u(0) = u_0$, find its temperature at any time t.

2.3 **(20 pts)** Consider a falling object of mass 5 kg that experiences the drag force, which is assumed to be proportional to the square of the velocity (denoted by v):

$$\dot{v} = \left[49^2 - v^2\right]/245.$$

(a) Determine an equilibrium solution.

(b) Plot a slope field for the given differential equation using one of your lovely software package. Provide the codes of your plot or state what resources did you use. Based on the direction field, determine the behavior of $v(t)$ as $t \to \infty$.

(c) Find the limiting velocity $v_\infty = \lim_{t\to\infty} v(t)$ if initially $v(0) = 0$, and determine the time that must elapse for the object to reach 98% of its limiting velocity.

(d) Find the time it takes the object to fall 300 m.

2.4 **(10 pts)** At a given level of effort, it is reasonable to assume that the rate at which fish are caught depends on the population $P(t)$: the more fish there are, the easier it is to catch them. Thus we assume that the rate at which fish are caught is given by $E\,P(t)$, where E is a positive constant, with units of 1/time, that measures the total effort made to harvest the given species of fish. To include this effect, the logistic equation is replaced by

$$\mathrm{d}P/\mathrm{d}t = r\,(1 - P/K)\,P - EP, \tag{i}$$

where r and K are positive constants. This equation is known as the **Schaefer model**.

(a) Show that if $E < r$, the there are two equilibrium points $P_1 = 0$ and $P_2 = K\,(1 - E/r) > 0$.

(b) Show that $P = P_1$ is unstable and $P = P_2$ is asymptotically stable. As a confirmation, you may want to draw a direction field for some numerical values of constants r, K, and E.

(c) A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort E and the asymptotically stable population P_2. Find Y as a function of the effort E: the graph of this function is known as the yield–effort curve.

(d) Determine E so as to maximize Y and thereby find the **maximum sustainable yield** Y_m.

2.5 (10 pts) Assuming that fish are caught at a constant rate h, its population is modeled by

$$\mathrm{d}P/\mathrm{d}t = r\,(1 - P/K)\,P - h. \tag{ii}$$

(a) If $h < rK/4$, show that Eq. (ii) has two equilibrium points q_1 and q_2 with $q_1 < q_2$; determine these points.

(b) Show that q_1 is unstable and q_2 is asymptotically stable.

(c) From a plot of the rate function $r\,(1 - P/K)\,P - h$ versus P, show that if the initial population $P(0) > q_1$, then $P(t) \mapsto q_2$ as $t \to \infty$, but that if $P(0) < q_1$, then $P(t)$ decreases as t increases. Note that $P \equiv 0$ is not an equilibrium point, so if $P(0) < q_1$, then extinction will be reached in a finite time.

(d) If $h > rK/4$, show that $P(t)$ decreases to zero as t increases, regardless of the value of $P(0)$.

(e) If $h = rK/4$, show that there is a single equilibrium point $P \equiv K/2$ and that this point is semistable. Thus the maximum sustainable yield is $h_m = rK/4$, corresponding to the equilibrium value $P \equiv K/2$. Observe that h_m has the same value as Y_m.

2.6 (10 pts) Solve the given differential equation of the form $x\,y' = y\,F(xy)$ by using transformation $v = xy$.

(a) $x\,y' = e^{xy} - y$; **(b)** $x\,y' = y/(xy + 1)$.

2.7 (10 pts) Solve the differential equation $y' = (4x + y - 5)^2$ by using appropriate transformation.

2.8 (20 pts) Solve the given differential equation with a homogeneous right-hand side function. Then determine an arbitrary constant that satisfies the auxiliary condition.

(a) $xy\,\mathrm{d}x + (x^2 + 3y^2)\,\mathrm{d}y = 0, \quad y(1) = 1;$

(b) $(y + \sqrt{x^2 + y^2})\,\mathrm{d}x - 2x\,\mathrm{d}y = 0, \quad y(1) = 0;$

(c) $(x - y)\,\mathrm{d}x + (3x + 2y)\,\mathrm{d}y = 0, \quad y(2) = 1;$

(d) $(y^2 + 3xy)\,\mathrm{d}x - 2x^2\,\mathrm{d}y = 0, \quad y(1) = 1.$