2.2 SINKS, SOURCES, AND SADDLES

We introduced the term "sink" in our discussion of one-dimensional maps to refer to a fixed point or periodic orbit that attracts an ϵ-neighborhood of initial values. A source is a fixed point that repels a neighborhood. These definitions make sense in higher-dimensional state spaces without alteration. In the plane, for example, the neighborhoods in question are disks (interiors of circles).

Definition 2.1 The **Euclidean length** of a vector $\mathbf{v} = (x_1, \ldots, x_m)$ in $\mathbb{R}^m$ is $|\mathbf{v}| = \sqrt{x_1^2 + \cdots + x_m^2}$. Let $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m$, and let ϵ be a positive number. The **ϵ-neighborhood** $N_\epsilon(\mathbf{p})$ is the set $\{\mathbf{v} \in \mathbb{R}^m : |\mathbf{v} - \mathbf{p}| < \epsilon\}$, the set of points within Euclidean distance ϵ of $\mathbf{p}$. We sometimes call $N_\epsilon(\mathbf{p})$ an **ϵ-disk** centered at $\mathbf{p}$.

Definition 2.2 Let $\mathbf{f}$ be a map on $\mathbb{R}^m$ and let $\mathbf{p}$ in $\mathbb{R}^m$ be a fixed point, that is, $\mathbf{f}(\mathbf{p}) = \mathbf{p}$. If there is an $\epsilon > 0$ such that for all $\mathbf{v}$ in the ϵ-neighborhood $N_\epsilon(\mathbf{p})$, $\lim_{k\to\infty} \mathbf{f}^k(\mathbf{v}) = \mathbf{p}$, then $\mathbf{p}$ is a **sink** or **attractor**. If there is an ϵ-neighborhood $N_\epsilon(\mathbf{p})$ such that each $\mathbf{v}$ in $N_\epsilon(\mathbf{p})$ except for $\mathbf{p}$ itself eventually maps outside of $N_\epsilon(\mathbf{p})$, then $\mathbf{p}$ is a **source** or **repeller**.

Figure 2.8 shows schematic views of a sink and a source for a two-dimensional map, along with a typical disk neighborhood and its image under the map. Along with the sink and source, a new type of fixed point is shown in Figure 2.8(c), which cannot occur in a one-dimensional state space. This type of fixed point, which we will call a saddle, has at least one attracting direction and at least one repelling direction. A saddle exhibits sensitive dependence on initial conditions, because of the neighboring initial conditions that escape along the repelling direction.

EXAMPLE 2.3

Consider the two-dimensional map

$$\mathbf{f}(x, y) = (-x^2 + 0.4y, x). \tag{2.14}$$

This is a version of the Hénon map considered earlier in this chapter, with the parameters set at $a = 0$ and $b = 0.4$.

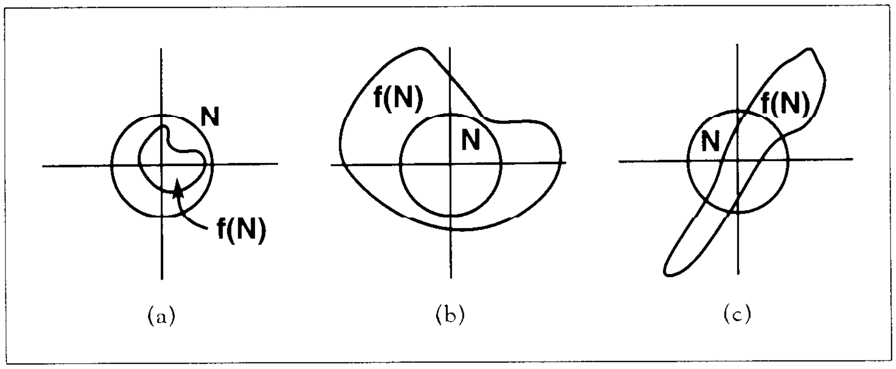

Figure 2.8 Local dynamics near a fixed point.
The origin is (a) a sink, (b) a source, and (c) a saddle. Shown is a disk N and its iterate under the map $\mathbf{f}$.

EXERCISE T2.2

Show that the map in (2.14) has exactly two fixed points, $(0, 0)$ and $(-0.6, -0.6)$.

Figure 2.9 shows the two fixed points. Around each is drawn a small disk N of radius 0.3. Also shown are the images $\mathbf{f}(N)$ and $\mathbf{f}^2(N)$ of each disk. The fixed point $(0, 0)$ is a sink, and the fixed point $(-0.6, -0.6)$ is a saddle. Each time the map is iterated, the disks shrink to 40% of their previous size. Therefore $\mathbf{f}(N)$ is 40% the size of N, and $\mathbf{f}^2(N)$ is 16% the size of N. We will explain the origin of these numbers in Remark 2.15.

Although saddles, as well as sources, are unstable fixed points (they are sensitive to initial conditions), they play surprising roles in the dynamics. In Figure 2.10(a), the basin of the sink $(0, 0)$ is shown in white. The entire square is the box $[-2, 2] \times [-2, 2]$, and the sink is the cross at the center. Not all of the white basin is shown: it has infinite area. The points in black diverge under iteration by $\mathbf{f}$; they are in the basin of infinity. You may wonder about the final disposition of the points along the boundary between the two basins. Do they go in or out? The answer is: neither. In Figure 2.10(b), the set of points that converge to the saddle $(-0.6, -0.6)$ is plotted, along with the saddle denoted by the cross. Although not an attractor, the saddle evidently plays a decisive role in determining which points go to which basin.

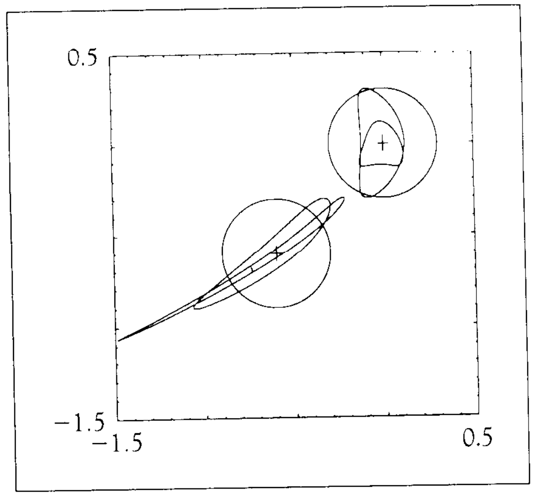

Figure 2.9 Local dynamics near fixed points of the Hénon map.
The crosses mark two fixed points of the Hénon map $\mathbf{f}$ with $a = 0$, $b = 0.4$, in the square $[-1.5, 0.5] \times [-1.5, 0.5]$. Around each fixed point a circle is drawn along with its two forward images under $\mathbf{f}$. On the left is a saddle: the images of the disk are becoming increasingly long and thin. On the right the images are shrinking, signifying a sink.

Attractors, as well as basins, can be more complicated than those shown in Figure 2.10. Consider the Hénon map (2.8) with $a = 2$ and $b = -0.3$. In Figure 2.11 the dark area is again the basin of infinity, while the white set is the basin for the two-piece attractor that looks like two hairpin curves.

Our goal in the next few sections is to find ways of identifying sinks, sources and saddles from the defining equations of the map. In Chapter 1 we found that the key to deciding the stability of a fixed point is the derivative at the point. Since the derivative determines the tangent line, or best linear approximation near the point, it determines the amount of shrinking/stretching in the vicinity of the point. The same mechanism is operating in higher dimensions. The action of the dynamics in the vicinity of a fixed point is governed by the best linear approximation to the map at the point. This best linear approximation is given by the Jacobian matrix, a matrix of partial derivatives calculated from the map. We will define the Jacobian matrix in Section 2.5. To find out what it can tell us, we need to fully understand linear maps first. For linear maps, the Jacobian matrix is equal to the map itself.

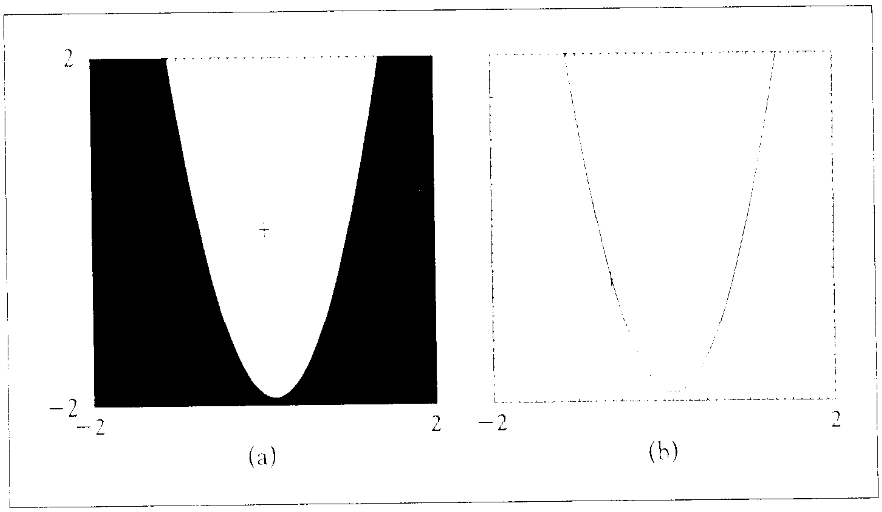

Figure 2.10 Basins of attraction for the Hénon map with $a = 0, b = 0.4$.
(a) The cross marks the fixed point $(0, 0)$. The basin of the fixed point $(0, 0)$ is shown in white; the points in black diverge to infinity. (b) The initial conditions that are on the boundary between the white and black don't converge to $(0, 0)$ or infinity; instead they converge to the saddle $(-0.6, -0.6)$, marked with a cross. This set of boundary points is the stable manifold of the saddle (to be discussed in Section 2.6).

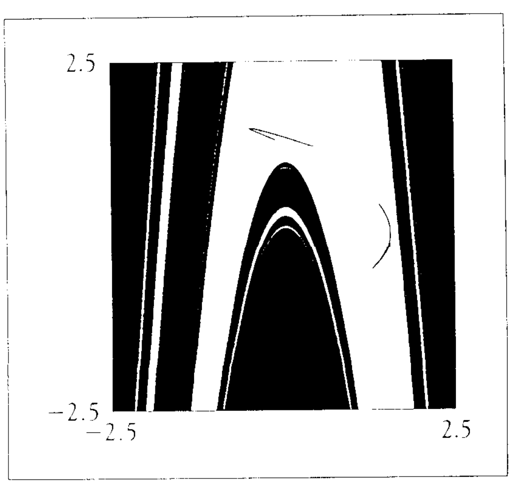

Figure 2.11 Attractors for the Hénon map with $a = 2$, $b = -0.3$.
Initial values in the white region are attracted to the hairpin attractor inside the white region. On each iteration, the points on one piece of the attractor map to the other piece. Orbits from initial values in the black region diverge to infinity.

2.3 LINEAR MAPS

The linear maps on $\mathbb{R}^2$ are those of the particularly simple form $\mathbf{v} \mapsto A\mathbf{v}$, where A is a 2×2 matrix:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}. \tag{2.15}$$

Definition 2.4 A map $A(\mathbf{v})$ from $\mathbb{R}^m$ to $\mathbb{R}^m$ is **linear** if for each $a, b \in \mathbb{R}$, and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$, $A(a\mathbf{v} + b\mathbf{w}) = aA(\mathbf{v}) + bA(\mathbf{w})$. Equivalently, a linear map $A(\mathbf{v})$ can be represented as multiplication by an $m \times m$ matrix.

Every linear map has a fixed point at the origin. This is analogous to the one-dimensional linear map $f(x) = ax$. The stability of the fixed point will be investigated the same way as in Chapter 1. If all of the points in a neighborhood of the fixed point $(0, 0)$ approach the fixed point when iterated by the map, we consider the fixed point to be an attractor.

In some cases the dynamics for a two-dimensional map resemble one-dimensional dynamics. Recall that λ is an **eigenvalue** of the matrix A if there is a nonzero vector $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

The vector $\mathbf{v}$ is called an **eigenvector**. Notice that if $\mathbf{v}_0$ is an eigenvector with eigenvalue λ, we can write down a special trajectory

$$\mathbf{v}_{n+1} = A\mathbf{v}_n$$

that satisfies

$$\mathbf{v}_1 = A\mathbf{v}_0 = \lambda\mathbf{v}_0$$
$$\mathbf{v}_2 = A\lambda\mathbf{v}_0 = \lambda A\mathbf{v}_0 = \lambda^2\mathbf{v}_0,$$

and in general $\mathbf{v}_n = \lambda^n\mathbf{v}_0$. Hence the map behaves like the one-dimensional map $x_{n+1} = \lambda x_n$.

We will begin by looking at three important examples of linear maps on $\mathbb{R}^2$. In fact, the three different types of 2×2 matrices we will encounter will be more than just good examples, they will be all possible examples, up to change of coordinates.

EXAMPLE 2.5

[Distinct real eigenvalues.] Let $\mathbf{v} = (x, y)$ denote a two-dimensional vector, and let $A(\mathbf{v})$ be the map on $\mathbb{R}^2$ defined by

$$A(x, y) = (ax, by).$$

Each input is a two-dimensional vector; so is each output. Any linear map can be represented by multiplication by a matrix, and following tradition we use A also to represent the matrix. Thus

$$A(\mathbf{v}) = A\mathbf{v} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.16}$$

The eigenvalues of the matrix A are a and b, with associated eigenvectors $(1, 0)$ and $(0, 1)$, respectively. For the purposes of this example, we will assume that they are not equal, although most of what we say now will not depend on that fact. Part of the importance of this example comes from the fact that a large class of linear maps can be expressed in the form (2.16), if the right coordinate system is used. For example, it is shown in Appendix A that any 2×2 matrix with distinct real eigenvalues takes this form when its eigenvectors are used to form the basis vectors of the coordinate system.

For the map in Example 2.5, the result of iterating the map n times is represented by the matrix

$$A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}. \tag{2.17}$$

The unit disk is mapped into an ellipse with semi-major axes of length $|a|^n$ along the x-axis and $|b|^n$ along the y-axis. An epsilon disk $N_\epsilon(0, 0)$ would become an ellipse with axes of length $\epsilon|a|^n$ and $\epsilon|b|^n$. For example, suppose that a and b are smaller than 1 in absolute value. Then this ellipse shrinks toward the origin as $n \to \infty$, so $(0, 0)$ is a sink for A. If $|a|, |b| > 1$, then the origin is a source.

On the other hand, if $|a| > 1 > |b|$, we see dynamical behavior that is not seen in one-dimensional maps. As n is increased, the ellipse grows in the x-direction and shrinks in the y-direction, essentially growing to look like the x-axis as $n \to \infty$. In Figure 2.12, we plot the unit disk and its two iterates under A where we set $a = 2$ and $b = 1/2$. In this case, the origin is neither a sink nor a source. If the ellipses formed by successive iterates of the map grow without bound along one direction and shrink to zero along another, we will call the origin a saddle.

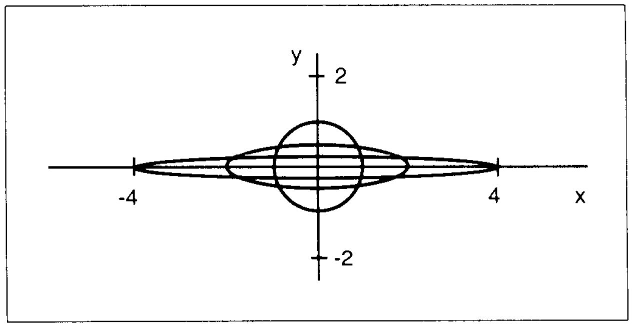

Figure 2.12 The unit disk and two images of the unit disk under a linear map. The origin is a saddle fixed point.

Points act as if they were moving along the surface of a saddle under the influence of gravity. A cowboy who spills coffee on his saddle will see it run toward the center along the front-to-back axis of the saddle (the y-axis in Figure 2.13) and run away from the center along the side-to-side axis (the x-axis in Figure 2.13). Presumably, a drop situated at the exact center of the saddle would stay there (our assumption is that the horse is not moving).

We see the same behavior for the iteration of points in Figure 2.14, which illustrates the linear map represented by the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}. \tag{2.18}$$

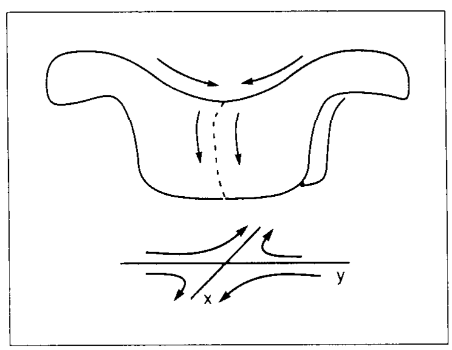

Figure 2.13 Dynamics near a saddle point.
Points in the vicinity of a saddle fixed point (here the origin in the xy-plane) move as if responding to the influence of gravity on a saddle.

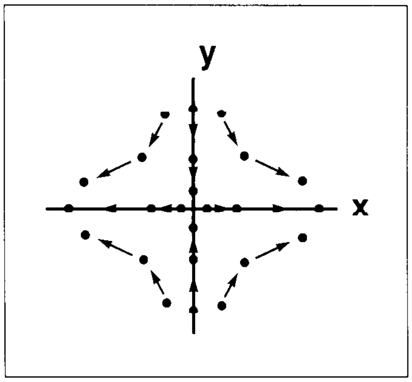

Figure 2.14 Saddle dynamics.
Successive images of points near a saddle fixed point are shown.

A typical point (x_0, y_0) maps to $(2x_0, \frac{1}{2}y_0)$, and then to $(4x_0, \frac{1}{4}y_0,)$, and so on. Notice that the product of the x- and y- coordinates is the constant quantity x_0y_0, so that orbits shown in Figure 2.14 traverse the hyperbola xy = constant = x_0y_0. More generally, for a linear map A on $\mathbb{R}^2$, the origin is a saddle if and only if iteration of the unit disk results in ellipses whose two axis lengths converge to zero and infinity, respectively.

A simplification can be made when analyzing small neighborhoods under linear maps. Because linearity implies $A(\mathbf{v}) = |\mathbf{v}|A(\frac{\mathbf{v}}{|\mathbf{v}|})$, the image of a vector $\mathbf{v}$ can be found by mapping the unit vector in the direction of $\mathbf{v}$, followed by scalar multiplication by the magnitude $|\mathbf{v}|$. The effect of the map on a small disk neighborhood of the origin is just a scaled-down version of the effect on the unit disk $N = N_1(0,0) = \{\mathbf{v} : |\mathbf{v}| < 1\}$. As a result we will often restrict our attention to the effect of the matrix on the unit disk. For example, the image of the unit disk centered at the origin under multiplication by any matrix is a filled ellipse centered at the origin. If the radius of the disk is r instead of 1, the resulting ellipse will also be changed precisely by a factor of r. (The semi-major axes will be changed by a factor of r.)

EXAMPLE 2.6

[Repeated eigenvalue.] For an example where the eigenvalues are not distinct, let

$$A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}. \qquad (2.19)$$

The eigenvalues of an upper triangular matrix are its diagonal entries, so the matrix has a repeated eigenvalue of a. Check that

$$A^n = a^{n-1}\begin{pmatrix} a & n \\ 0 & a \end{pmatrix}. \tag{2.20}$$

Therefore the effect of A^n on vectors is

$$A^n\begin{pmatrix} x \\ y \end{pmatrix} = a^{n-1}\begin{pmatrix} ax + ny \\ ay \end{pmatrix}. \tag{2.21}$$

✎ EXERCISE T2.3

(a) Verify equation (2.20). (b) Use equation (2.21) to show that the fixed point $(0,0)$ is a sink if $|a| < 1$ and a source if $|a| > 1$.

EXAMPLE 2.7

[Complex eigenvalues.] Let

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{2.22}$$

This matrix has no real eigenvalues. The eigenvalues of this matrix are $a - bi$ and $a + bi$, where $i = \sqrt{-1}$. The corresponding eigenvectors are $(1, -i)$ and $(1, i)$, respectively. Fortunately, this information can be interpreted in terms of real vectors. A more intuitive way to look at this matrix follows from multiplying and dividing by $r = \sqrt{a^2 + b^2}$. Then

$$A = r\begin{pmatrix} a/r & -b/r \\ b/r & a/r \end{pmatrix} = \sqrt{a^2 + b^2}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2.23}$$

Here we used the fact that any pair of numbers c, s such that $c^2 + s^2 = 1$ can be written as $c = \cos\theta$ and $s = \sin\theta$ for some angle θ. The angle θ can be identified as $\theta = \arctan(b/a)$. It is now clear that multiplication by this matrix rotates points about the origin by an angle θ, and multiplies the distances by $\sqrt{a^2 + b^2}$. Therefore it is a combination of a rotation and a dilation.

✎ EXERCISE T2.4

Verify that multiplication by A rotates a vector by $\arctan(b/a)$ and stretches by a factor of $\sqrt{a^2 + b^2}$.