Therefore the map has this matrix representation in some coordinate system. Referring to Example 2.5, we see that the origin is a sink if $|a|, |b| < 1$ and a source if $|a|, |b| > 1$. The same analysis works for matrices with repeated eigenvalues, or a pair of complex eigenvalues. Summing up, we have proved the $m = 2$ version of the following theorem.

Theorem 2.8 *Let $A(\mathbf{v})$ be a linear map on $\mathbb{R}^m$, which is represented by the matrix A (in some coordinate system). Then*

1. *The origin is a sink if all eigenvalues of A are smaller than one in absolute value;*
2. *The origin is a source if all eigenvalues of A are larger than one in absolute value.*

In dimensions two and greater, we must also consider linear maps of mixed stability, i.e., those for which the origin is a saddle.

Definition 2.9 Let A be a linear map on $\mathbb{R}^m$. We say A is **hyperbolic** if A has no eigenvalues of absolute value one. If a hyperbolic map A has at least one eigenvalue of absolute value greater than one and at least one eigenvalue of absolute value smaller than one, then the origin is called a **saddle**.

Thus there are three types of hyperbolic maps: ones for which the origin is a sink, ones for which the origin is a source, and ones for which the origin is a saddle. Hyperbolic linear maps are important objects of study because they have well-defined expanding and contracting directions.

2.5 NONLINEAR MAPS AND THE JACOBIAN MATRIX

So far we have discussed linear maps, which always have a fixed point at the origin. We now want to discuss nonlinear maps, and in particular how to determine the stability of fixed points.

Our treatment of stability in Chapter 1 is relevant to this case. Theorem 1.5 showed that whether a fixed point of a one-dimensional nonlinear map is a sink or source depends on its "linearization", or linear part, near the fixed point. In the one-dimensional case the linearization is given by the derivative at the fixed point. If p is a fixed point and h is a small number, then the change in the output of the map at $p + h$, compared to the output at p, is well approximated by the linear map $L(h) = Kh$, where K is the constant number $f'(p)$. In other words,

$$f(p + h) \approx f(p) + hf'(p). \tag{2.25}$$

Our proof of Theorem 1.5 was based on the fact that the error in this approximation was of size proportional to h^2. This can be made as small as desired by restricting attention to sufficiently small h. If $|f'(p)| < 1$, the fixed point p is a sink, and if $|f'(p)| > 1$, it is a source. The situation is very similar for nonlinear maps in higher dimensions. The place of the derivative in the above discussion is taken by a matrix.

Definition 2.10 Let $\mathbf{f} = (f_1, f_2, \ldots, f_m)$ be a map on $\mathbb{R}^m$, and let $\mathbf{p} \in \mathbb{R}^m$. The **Jacobian matrix** of $\mathbf{f}$ at $\mathbf{p}$, denoted $\mathbf{Df}(\mathbf{p})$, is the matrix

$$\mathbf{Df}(\mathbf{p}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{p}) & \cdots & \frac{\partial f_1}{\partial x_m}(\mathbf{p}) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{p}) & \cdots & \frac{\partial f_m}{\partial x_m}(\mathbf{p}) \end{pmatrix}$$

of partial derivatives evaluated at $\mathbf{p}$.

Given a vector $\mathbf{p}$ and a small vector $\mathbf{h}$, the increment in $\mathbf{f}$ due to $\mathbf{h}$ is approximated by the Jacobian matrix times the vector $\mathbf{h}$:

$$\mathbf{f}(\mathbf{p} + \mathbf{h}) - \mathbf{f}(\mathbf{p}) \approx \mathbf{Df}(\mathbf{p}) \cdot \mathbf{h}, \tag{2.26}$$

where again the error in the approximation is proportional to $|\mathbf{h}|^2$ for small $\mathbf{h}$. If we assume that $\mathbf{f}(\mathbf{p}) = \mathbf{p}$, then for a small change $\mathbf{h}$, the map moves $\mathbf{p} + \mathbf{h}$ approximately $\mathbf{Df}(\mathbf{p}) \cdot \mathbf{h}$ away from $\mathbf{p}$. That is, $\mathbf{f}$ magnifies a small change $\mathbf{h}$ in input to a change $\mathbf{Df}(\mathbf{p}) \cdot \mathbf{h}$ in output.

As long as this deviation remains small (so that $|\mathbf{h}|^2$ is negligible and our approximation is valid), the action of the map near $\mathbf{p}$ is essentially the same as the linear map $\mathbf{h} \mapsto A\mathbf{h}$, where $A = \mathbf{Df}(\mathbf{p})$, with fixed point $\mathbf{h} = \mathbf{0}$. Small disk neighborhoods centered at $\mathbf{h} = 0$ (corresponding to disks around $\mathbf{p}$) map to regions approximated by ellipses whose axes are determined by A. In that case, we can appeal to Theorem 2.8 for information about linear stability for higher-dimensional maps in order to understand the nonlinear case.

The following theorem is an extension of Theorems 1.5 and 2.8 to higher dimensional nonlinear maps. It determines the stability of a map at a fixed point based on the Jacobian matrix at that point. The proof is omitted.

Theorem 2.11 *Let* $\mathbf{f}$ *be a map on* $\mathbb{R}^m$, *and assume* $\mathbf{f}(\mathbf{p}) = \mathbf{p}$.

1. *If the magnitude of each eigenvalue of* $\mathbf{Df}(\mathbf{p})$ *is less than 1, then* $\mathbf{p}$ *is a sink.*
2. *If the magnitude of each eigenvalue of* $\mathbf{Df}(\mathbf{p})$ *is greater than 1, then* $\mathbf{p}$ *is a source.*

Just as linear maps of $\mathbb{R}^m$ for $m > 1$ can have some directions in which orbits diverge from 0 and some in which orbits converge to 0, so fixed points of nonlinear maps can attract points in some directions and repel points in others.

Definition 2.12 Let $\mathbf{f}$ be a map on $\mathbb{R}^m$, $m \geq 1$. Assume that $\mathbf{f}(\mathbf{p}) = \mathbf{p}$. Then the fixed point $\mathbf{p}$ is called **hyperbolic** if none of the eigenvalues of $\mathbf{Df}(\mathbf{p})$ has magnitude 1. If $\mathbf{p}$ is hyperbolic and if at least one eigenvalue of $\mathbf{Df}(\mathbf{p})$ has magnitude greater than 1 and at least one eigenvalue has magnitude less than 1, then $\mathbf{p}$ is called a **saddle**. (For a periodic point of period k, replace $\mathbf{f}$ by $\mathbf{f}^k$.)

Saddles are unstable. If even one eigenvalue of $\mathbf{Df}(\mathbf{p})$ has magnitude greater than 1, then $\mathbf{p}$ is unstable in the sense previously described: Almost any perturbation of the orbit away from the fixed point will be magnified under iteration. In a small epsilon neighborhood of $\mathbf{p}$, $\mathbf{f}$ behaves very much like a linear map with an eigenvalue that has magnitude greater than 1; that is, the orbits of most points near $\mathbf{p}$ diverge from $\mathbf{p}$.

EXAMPLE 2.13

The Hénon map

$$\mathbf{f}_{a,b}(x, y) = (a - x^2 + by, x), \tag{2.27}$$

where a and b are constants, has at most two fixed points. Setting $a = 0$ and $b = 0.4$, $\mathbf{f}$ has the two fixed points $(0, 0)$ and $(-0.6, -0.6)$. The Jacobian matrix $\mathbf{Df}$ is

$$\mathbf{Df}(x, y) = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix}. \tag{2.28}$$

Evaluated at $(0, 0)$, the Jacobian matrix is

$$\mathbf{Df}(0, 0) = \begin{pmatrix} 0 & 0.4 \\ 1 & 0 \end{pmatrix},$$

with eigenvalues $\pm\sqrt{0.4}$, approximately equal to 0.632 and -0.632. Evaluated at $(-0.6, -0.6)$, the Jacobian is

$$\mathbf{Df}(-0.6, -0.6) = \begin{pmatrix} 1.2 & 0.4 \\ 1 & 0 \end{pmatrix},$$

with eigenvalues approximately equal to 1.472 and −0.272. Thus (0, 0) is a sink and (−0.6, −0.6) is a saddle.

For the parameter values $a = 0.43, b = 0.4$, there is a period-two orbit for the map. Check that $\{(0.7, -0.1), (-0.1, 0.7)\}$ is such an orbit. In order to check the stability of this orbit, we need to compute the Jacobian matrix of $\mathbf{f}^2$ evaluated at (0.7, −0.1). Because of the chain rule, we can do this without explicitly forming $\mathbf{f}^2$, since $\mathbf{Df}^2(x) = \mathbf{Df}(\mathbf{f}(x)) \cdot \mathbf{Df}(x)$. We compute

$$\begin{aligned}
\mathbf{Df}^2((0.7, -0.1)) &= \mathbf{Df}((-0.1, 0.7)) \cdot \mathbf{Df}((0.7, -0.1)) \\
&= \begin{pmatrix} -2(-0.1) & 0.4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2(0.7) & 0.4 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0.12 & 0.08 \\ -1.4 & 0.4 \end{pmatrix}.
\end{aligned}$$

The eigenvalues of this Jacobian matrix are approximately $0.26 \pm 0.30i$, which are complex numbers of magnitude ≈ 0.4, so the period-two orbit is a sink.

Note that the same eigenvalues are obtained by evaluating

$$\mathbf{Df}^2((-0.1, 0.7)) = \mathbf{Df}((0.7, -0.1)) \cdot \mathbf{Df}((-0.1, 0.7)),$$

which means that stability is a property of the periodic orbit as a whole, not of the individual points of the orbit. This is true because the eigenvalues of a product AB of two matrices are identical to the eigenvalues of BA, as shown in the Appendix A. This result compares with (1.4) of Chapter 1.

Remark 2.14 For a map on $\mathbb{R}^m$, there is a more general statement of this fact. Assume there is a periodic orbit $\{\mathbf{p}_1, \ldots, \mathbf{p}_k\}$ of period k. By Lemma A.2 of Appendix A, the set of eigenvalues of a product of several matrices is unchanged under a cyclic permutation of the order of the product. Using the chain rule,

$$\mathbf{Df}^k(\mathbf{p}_1) = \mathbf{Df}(\mathbf{p}_k) \cdot \mathbf{Df}(\mathbf{p}_{k-1}) \cdots \mathbf{Df}(\mathbf{p}_1). \tag{2.29}$$

The eigenvalues of the $m \times m$ Jacobian matrix evaluated at $\mathbf{p}_1$, $\mathbf{Df}^k(\mathbf{p}_1)$, will determine the stability of the period-k orbit. But one should also be able to determine the stability by examining the eigenvalues of $\mathbf{Df}^k(\mathbf{p}_r)$, where $\mathbf{p}_r$ is one of the other points in the periodic orbit. Applying the chain rule as above, we find that

$$\mathbf{Df}^k(\mathbf{p}_r) = \mathbf{Df}(\mathbf{p}_{r-1}) \cdot \mathbf{Df}(\mathbf{p}_{r-2}) \cdots \mathbf{Df}(\mathbf{p}_1) \cdot \mathbf{Df}(\mathbf{p}_k) \cdots \mathbf{Df}(\mathbf{p}_r). \tag{2.30}$$

According to Lemma A.2, the eigenvalues of (2.29) and (2.30) are identical. This guarantees that the eigenvalues are shared by the periodic orbit, and can be measured by multiplying together the k Jacobian matrices starting at any of the k points.

A more systematic study can be made of the fixed points and period-two points of the Hénon map. Let the parameters a and b be arbitrary. Then all fixed points satisfy

$$\begin{aligned} x &= a - x^2 + by \\ y &= x, \end{aligned} \tag{2.31}$$

which is equivalent to the equation $x = a - x^2 + bx$, or

$$x^2 + (1-b)x - a = 0. \tag{2.32}$$

Using the quadratic formula, we see that fixed points exist as long as

$$4a > -(1-b)^2 \tag{2.33}$$

If (2.33) is satisfied, there are exactly two fixed points, whose x-coordinates are found from the quadratic formula and whose y-coordinate is the same as the x-coordinate.

To look for period-two points, set $(x, y) = \mathbf{f}^2(x, y)$:

$$\begin{aligned} x &= a - (a - x^2 + by)^2 + bx \\ y &= a - x^2 + by. \end{aligned} \tag{2.34}$$

Solving the second equation for y and substituting into the first, we get an equation for the x-coordinate of a period-two point:

$$\begin{aligned} 0 &= (x^2 - a)^2 + (1-b)^3 x - (1-b)^2 a \\ &= (x^2 - (1-b)x - a + (1-b)^2)(x^2 + (1-b)x - a). \end{aligned} \tag{2.35}$$

We recognize the factor on the right from Equation (2.32): Zeros of it correspond to fixed points of $\mathbf{f}$, which are also fixed points of $\mathbf{f}^2$. In fact, it was the knowledge that (2.32) must be a factor which was the trick that allowed us to write (2.35) in factored form. The period-two orbit is given by the zeros of the left factor, if they exist.

✎ **EXERCISE T2.5**

Prove that the Hénon map has a period-two orbit if and only if $4a > 3(1-b)^2$.

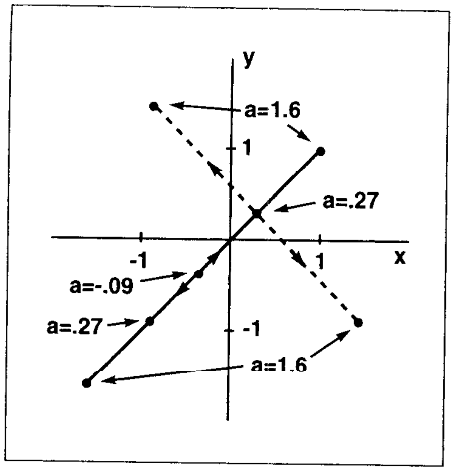

Figure 2.15 Fixed points and period-two points for the Hénon map with b fixed at 0.4.
The solid line denotes the trails of the two fixed points as a moves from -0.09, where the two fixed points are created together, to 1.6 where they have moved quite far apart. The fixed point that moves diagonally upward is attracting for $-0.09 < a < 0.27$; the other is a saddle. The dashed line follows the period-two orbit from its creation when $a = 0.27$, at the site of the (previously) attracting fixed point, to $a = 1.6$.

Figure 2.15 shows the fixed points and period-two points of the Hénon map for $b = .4$ and for various values of a. We understand why the fixed points lie along the diagonal line $y = x$, but why do the period-two orbits lie along a line, as shown in Figure 2.15?

✎ EXERCISE T2.6

(a) If (x_1, y_1) and (x_2, y_2) are the two fixed points of the Hénon map (2.27) with some fixed parameters a and b, show that $x_1 - y_1 = x_2 - y_2 = 0$ and $x_1 + x_2 = y_1 + y_2 = b - 1$.

(b) If $\{(x_1, y_1), (x_2, y_2)\}$ is the period-two orbit, show that $x_1 + y_1 = x_2 + y_2 = x_1 + x_2 = y_1 + y_2 = 1 - b$. In particular the period-two orbit lies along the line $x + y = 1 - b$, as seen in Figure 2.15.

Figure 2.16 shows a bifurcation diagram for the Hénon map for the case $b = 0.4$. For each fixed value $0 \le a \le 1.25$ along the horizontal axis, the x-coordinates of the attracting set are plotted vertically. The information in Figure

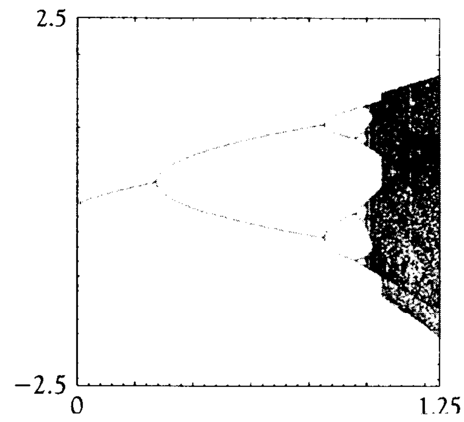

Figure 2.16 Bifurcation diagram for the Hénon map, $b = 0.4$.
Each vertical slice shows the projection onto the x-axis of an attractor for the map for a fixed value of the parameter a.

2.15 is recapitulated here. At $a = 0.27$, a period-doubling bifurcation occurs, when the fixed point loses stability and a period-two orbit is born. The period-two orbit is a sink until $a = 0.85$, when it too doubles its period. In the next exercise, you will be asked to use the equations we developed here to verify some of these facts.

✎ EXERCISE T2.7

Set $b = 0.4$.

(a) Prove that for $-0.09 < a < 0.27$, the Hénon map $\mathbf{f}$ has one sink fixed point and one saddle fixed point.

(b) Find the largest magnitude eigenvalue of the Jacobian matrix at the first fixed point when $a = 0.27$. Explain the loss of stability of the sink.

(c) Prove that for $0.27 < a < 0.85$, $\mathbf{f}$ has a period-two sink.

(d) Find the largest magnitude eigenvalue of $\mathbf{Df}^2$, the Jacobian of $\mathbf{f}^2$ at the period-two orbit, when $a = 0.85$.

For $b = 0.4$ and $a > 0.85$, the attractors of the Hénon map become more complex. When the period-two orbit becomes unstable, it is immediately replaced with an attracting period-four orbit, then a period-eight orbit, etc. Figure 2.17

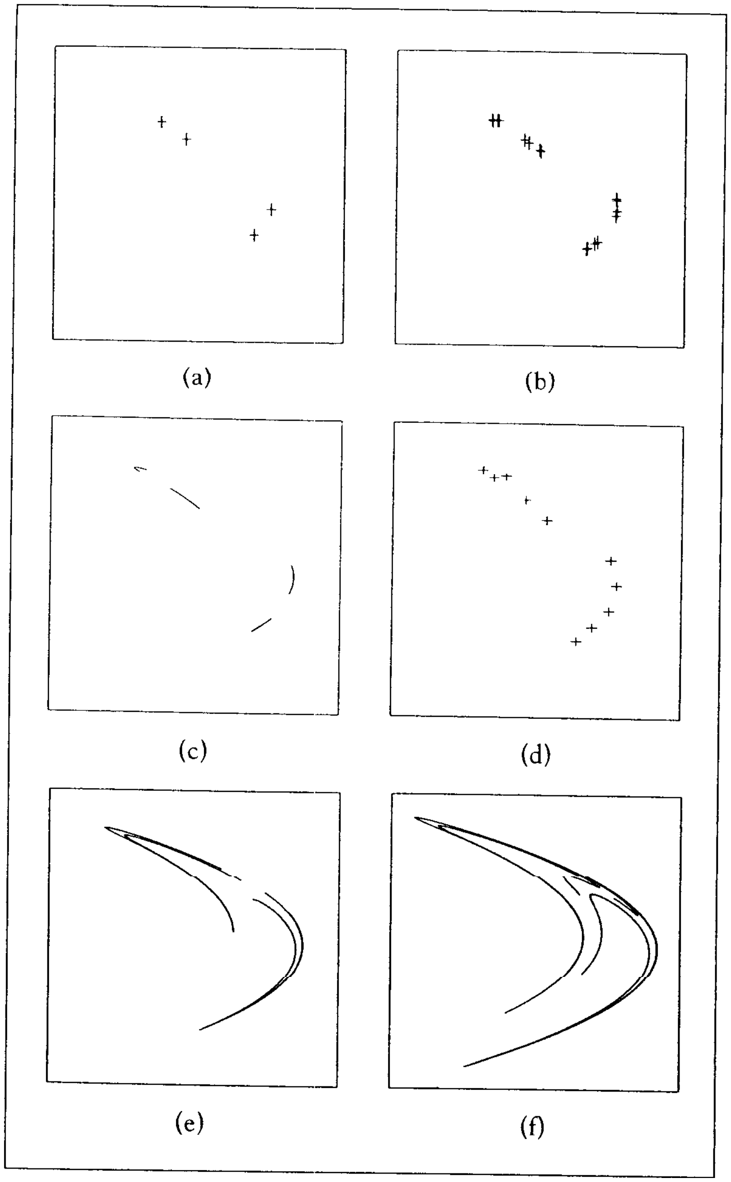

Figure 2.17 Attractors for the Hénon map with $b = 0.4$.
Each panel displays a single attracting orbit for a particular value of the parameter a. (a) $a = 0.9$, period 4 sink. (b) $a = 0.988$, period 16 sink. (c) $a = 1.0$, four-piece attractor. (d) $a = 1.0293$, period-ten sink. (e) $a = 1.045$, two-piece attractor. The points of an orbit alternate between the pieces. (f) $a = 1.2$, two pieces have merged to form one-piece attractor.

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shows a number of these attractors. An example is the "period-ten window" at $a = 1.0293$, barely detectable as a vertical white gap in Figure 2.16.

➪ COMPUTER EXPERIMENT 2.2

Make a bifurcation diagram like Figure 2.16, but for $b = -0.3$, and for $0 \le a \le 2.2$. For each a, choose the initial point $(0, 0)$ and calculate its orbit. Plot the x-coordinates of the orbit, starting with iterate 101 (to allow time for the orbit to approximately settle down to the attracting orbit). Questions to answer: Does the resulting bifurcation diagram depend on the choice of initial point? How is the picture different if the y-coordinates are plotted instead?

Periodic points are the key to many of the properties of a map. For example, trajectories often converge to a periodic sink. Periodic saddles and sources, on the other hand, do not attract open neighborhoods of initial values as sinks do, but are important in their own ways, as will be seen in later chapters.

Remark 2.15 The theme of this section has been the use of the Jacobian matrix for determining stability of periodic orbits of nonlinear maps, in the way that the map matrix itself is used for linear maps. There are other important uses for the Jacobian matrix. The magnitude of its determinant measures the transformation of areas for nonlinear maps, at least locally.

For example, consider the Hénon map (2.27). The determinant of the Jacobian matrix (2.28) is fixed at $-b$ for all $\mathbf{v}$. For the case $a = 0, b = 0.4$, the map $\mathbf{f}$ transforms area near each point $\mathbf{v}$ at the rate $|\det(\mathbf{Df}(\mathbf{v}))| = |-b| = 0.4$. Each plane region is transformed by $\mathbf{f}$ into a region that is 40% of its original size. The circle around each fixed point in Figure 2.9, for example, has forward images which are $.4 = 40\%$ and $(.4)^2 = .16 = 16\%$, respectively.

Most of the plane maps we will deal with are invertible, meaning that their inverses exist.

Definition 2.16 A map $\mathbf{f}$ on $\mathbb{R}^m$ is **one-to-one** if $\mathbf{f}(\mathbf{v}_1) = \mathbf{f}(\mathbf{v}_2)$ implies $\mathbf{v}_1 = \mathbf{v}_2$.

Recall that functions are well-defined by definition, i.e. $\mathbf{v}_1 = \mathbf{v}_2$ implies $\mathbf{f}(\mathbf{v}_1) = \mathbf{f}(\mathbf{v}_2)$. Two points do not get mapped together under a one-to-one map. It follows that if $\mathbf{f}$ is a one-to-one map, then its **inverse map** $\mathbf{f}^{-1}$ is a function. The

INVERSE MAPS

A function is a uniquely-defined assignment of a range point for each domain point. (If the domain and range are the same set, we call the function a map.) Several domain points may map to the same range point. For $\mathbf{f}_1(x, y) = (x^2, y^2)$, the points $(2, 2)$, $(2, -2)$, $(-2, 2)$ and $(-2, -2)$ all map to $(4, 4)$. On the other hand, for $\mathbf{f}_2(x, y) = (x^3, y^3)$, this never happens. A point (a, b) is the image of $(a^{1/3}, b^{1/3})$ only. Thus $\mathbf{f}_2$ is a one-to-one map, and $\mathbf{f}_1$ is not.

An inverse map $\mathbf{f}^{-1}$ automatically exists for any one-to-one map $\mathbf{f}$. The domain of $\mathbf{f}^{-1}$ is the image of $\mathbf{f}$. For the example $\mathbf{f}_2(x, y) = (x^3, y^3)$, the inverse is $\mathbf{f}_2^{-1}(x, y) = (x^{1/3}, y^{1/3})$.

To compute an inverse map, set $\mathbf{v}_1 = \mathbf{f}(\mathbf{v})$ and solve for $\mathbf{v}$ in terms of $\mathbf{v}_1$. We demonstrate using $\mathbf{f}(x, y) = (x + 2y, x^3)$. Set

$$x_1 = x + 2y$$
$$y_1 = x^3$$

and solve for x and y. The result is

$$x = y_1^{1/3}$$
$$y = (x_1 - y_1^{1/3})/2,$$

so that the inverse map is $\mathbf{f}(x, y) = (y^{1/3}, (x_1 - y_1^{1/3})/2)$.

inverse map is characterized by the fact that $\mathbf{f}(\mathbf{v}) = \mathbf{w}$ if and only if $\mathbf{v} = \mathbf{f}^{-1}(\mathbf{w})$. Because one-to-one implies the existence of an inverse, a one-to-one map is also called an **invertible** map.

✎ EXERCISE T2.8

Show that the Hénon map (2.27) with $b \neq 0$ is invertible by finding a formula for the inverse. Is the map one-to-one if $b = 0$?