



Fractional spectral collocation methods for linear and nonlinear variable order FPDEs

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ABSTRACT

While several high-order methods have been developed for fractional PDEs (FPDEs) with fixed order, there are no such methods for FPDEs with field-variable order. These equations allow multiphysics simulations seamlessly, e.g. from diffusion to sub-diffusion or from wave dynamics transitioning to diffusion, by simply varying the fractional order as a function of space or time. We develop an exponentially accurate fractional spectral collocation method for solving linear/nonlinear FPDEs with field-variable order. Following the spectral theory, developed in [1] for fractional Sturm–Liouville eigenproblems, we introduce a new family of interpolants, called *left-/right-sided* and *central fractional Lagrange interpolants*. We employ the fractional derivatives of (left-/right-sided) Riemann–Liouville and Riesz type and obtain the corresponding fractional differentiation matrices by collocating the field-variable fractional orders. We solve several FPDEs including time- and space-fractional advection–equation, time- and space-fractional advection–diffusion equation, and finally the space-fractional Burgers' equation to demonstrate the performance of the method. In addition, we develop a spectral penalty method for enforcing inhomogeneous initial conditions. Our numerical results confirm the exponential-like convergence of the proposed fractional collocation methods.

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1. Introduction

The theory of fractional differential operators generalizes the notion of standard operators of integer orders to fractional orders. Such differential operators appear in modeling diverse physical problems involving e.g., porous or fractured media [2], viscoelastic materials [3], viscous fluid flows subject to wall-friction effects [4–6], bioengineering applications [7], and *anomalous* transport [8–10]. The notion of fractional derivatives has been rapidly extended to a variety of fractional partial differential equations (FPDEs) such as fractional Burgers' equation [11], Fokker–Planck equation [12], and advection–diffusion equation [13].

Recently, it has been demonstrated that in many dynamic processes, the underlying differential operators not only appear as fractional, but they also possess a dynamic nature in a sense that their order is *field-variable*, which may vary in time and/or space. For instance, several classes of random processes with variable-order fractional transition probability densities on unbounded domains have been studied in [14,15]. Moreover, the notion of *variable-order* fractional calculus has been used to dynamic modeling of heterogeneous physical systems. Examples are linear and nonlinear oscillators with viscoelastic damping by Coimbra [16], processing of geographical data using variable-order derivatives by Cooper and Cowan [17],

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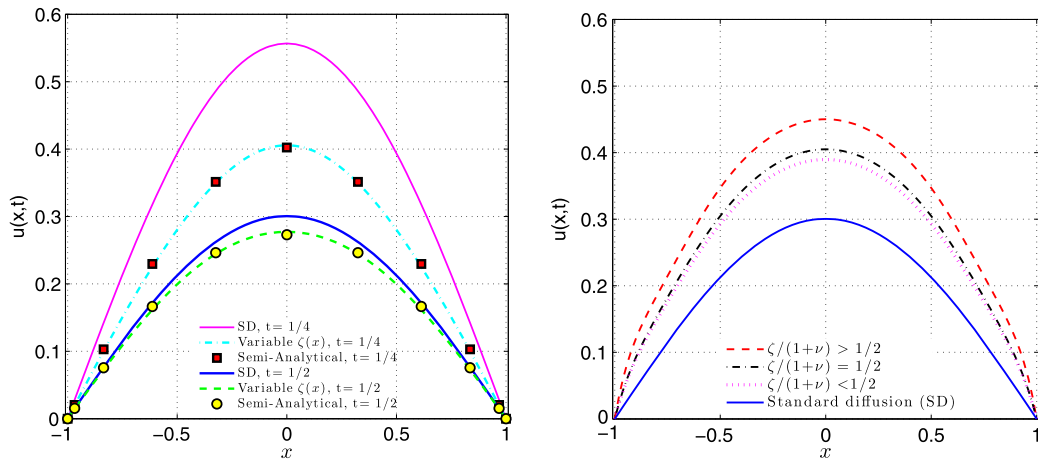


Fig. 1. Variable-order (left) versus fixed-order diffusion (right). The initial condition is $u(x, 0) = 1 - x^2$ and the solutions are obtained at $t = \frac{1}{2}$, where the space-fractional order is $(1 + \nu) = 1.99$. While the fixed-order cases on the right plot exhibit the expected *sub-diffusion* process compared to the standard diffusion (SD) problem (i.e., when $\zeta = 1$ and $1 + \nu = 2$), the variable-order test-case when $\zeta(x) = 1/(3|x| + 11/10)$ on the left plot exhibit, surprisingly, a *super-diffusion* behavior.

signature verification through variable/adaptive fractional order differentiators by Tseng [18], constitutive laws in viscoelastic continuum mechanics by Ramirez et al. [19], modeling of diffusive-convective effects on the oscillatory flows [20], anomalous diffusion problems by Sun et al. [21], fractional advection-diffusion problem by Chen et al. [22,23], mobile-immobile advection-dispersion model by Zhang et al. [24], and chloride ions sub-diffusion in concrete structures by Chen et al. [25]. This approach opens up great opportunities for modeling and simulation of multiphysics phenomena, e.g. seamless transition from wave propagation to diffusion, or from local to non-local dynamics.

Such an extension from fixed-order to variable-order operators provides an invaluable prospect in modeling complex phenomena, whose behavior otherwise may not be properly understood. For instance, in Fig. 1(right), we investigate the decaying solution to the following time- and space-fractional diffusion problem

$${}_0^C \mathcal{D}_t^{\zeta(x)} u = \frac{\partial^{1+\nu} u}{\partial |x|^{1+\nu}}, \quad (1)$$

subject to $u(x, 0) = (1 - x^2)$ and $u(\pm 1, t) = 0$, in the absence of any external forces. In (1), ${}_0^C \mathcal{D}_t^{\zeta}(\cdot)$ and $\frac{\partial^{1+\nu}(\cdot)}{\partial |x|^{1+\nu}}$ are fractional derivatives of Caputo and Riesz type, see Section 2. Here, we set ζ and $\nu \in (0, 1)$, highlighting the anomalous *sub-diffusion* character of the problem, compared to the standard diffusion denoted by SD, i.e., when $\zeta = \nu = 1$. We demonstrate the corresponding sub-diffusive behavior in (1) by taking different but *fixed* values of ζ and ν and plotting the results in Fig. 1(right) at $t = 1/2$. As expected, the corresponding curves all lag behind the standard diffusion. Surprisingly, when we allow the temporal order ζ to vary across the domain, as shown in Fig. 1(left) at $t = 1/4$ and $t = 1/2$, we observe a *super-diffusive* behavior even for $\zeta(x) \in (0, 1)$; here, $\zeta(x) = 1/(3|x| + 11/10)$. In addition, we notice a more pronounced super-diffusion at earlier times in Fig. 1(left). However, after a long time, the variable-order case becomes a sub-diffusion process.

The numerical approximation of variable-order FPDEs has been mostly developed using finite-difference methods (FDMs) (see [26–30] and references therein). Although easier to implement, the main challenge in FDM schemes is their limited accuracy interwoven with their inherent *local* character, while fractional derivatives are essentially *global* (nonlocal) differential operators. Hence, global schemes such as *Spectral Methods* (SM) may be more appropriate for discretizing fractional operators.

The idea of developing spectral methods for *fixed order* FODEs/FPDEs has received great attention over the last decade. Lin and Xu [31] developed a hybrid scheme for time-fractional diffusion problem; they also developed a time-space SM for time-fractional diffusion [32,33]. Khader [34] proposed a Chebyshev collocation method for a space-fractional diffusion equation; while Piret and Hanert developed a radial basis function method for fractional diffusion equations [35]. Moreover, a Chebyshev spectral method [36], a Legendre spectral method [37], and an adaptive pseudospectral method [38] were proposed for solving fractional boundary value problems. In addition, generalized Laguerre spectral algorithms and Legendre spectral Galerkin method were developed by Baleanu et al. [39] and by Bhrawy and Alghamdi [40] for fractional initial value problems. Recently, Deng and Hesthaven [41] developed local discontinuous Galerkin methods for fractional diffusion equations, and Xu and Hesthaven [42] developed a stable multi-domain spectral penalty method for FPDEs. In all the aforementioned spectral methods, *polynomial* bases have been employed.

Recently, Zayernouri and Karniadakis [43,44] and Zayernouri et al. [45] developed spectrally accurate Petrov-Galerkin spectral and spectral element methods for non-delay and delay fractional differential equations in addition to the fractional advection equation, where they employed a new family of fractional bases, called *Jacobi Polyfractonomials*.

They introduced these polyfractonomials as the eigenfunctions of fractional Sturm–Liouville problems in [1], explicitly given as

$$^{(1)}\mathcal{P}_n^{\alpha,\beta,\mu}(\xi) = (1+\xi)^{-\beta+\mu-1} P_{n-1}^{\alpha-\mu+1,-\beta+\mu-1}(\xi), \quad \xi \in [-1, 1], \quad (2)$$

with $\mu \in (0, 1)$, $-1 \leq \alpha < 2 - \mu$, and $-1 \leq \beta < \mu - 1$, representing the eigenfunctions of the *singular* FSLP of first kind (SFSLP-I), and

$$^{(2)}\mathcal{P}_n^{\alpha,\beta,\mu}(\xi) = (1-\xi)^{-\alpha+\mu-1} P_{n-1}^{-\alpha+\mu-1,\beta-\mu+1}(\xi), \quad \xi \in [-1, 1], \quad (3)$$

where $-1 < \alpha < \mu - 1$ and $-1 < \beta < 2 - \mu$, and $\mu \in (0, 1)$, denoting the eigenfunctions of the *singular* FSLP of *second* kind (SFSLP-II). Moreover, they employed these fractional bases to introduce a new class of fractional interpolants to develop efficient and spectrally accurate collocation methods in [46] for a variety of FODEs and FPDEs including multi-term FPDEs and the nonlinear space-fractional Burgers' equation. Employing the Jacobi polyfractonomials, Zayernouri and Karniadakis have recently developed a unified Petrov–Galerkin spectral method along with a unified fast solver in [47] that efficiently treats the whole family of elliptic, parabolic, and hyperbolic FPDEs in high-dimensions with spectral accuracy and in a unified fashion.

To the best of our knowledge, no high-order or spectral methods have been developed for variable-order FPDEs up to date. The main contribution of the present study is the development of a new class of spectrally accurate fractional spectral collocation method for solving linear and nonlinear field-variable order in-time and in-space FPDEs. To this end, we introduce the *left/right-sided* fractional Lagrange interpolants (FLIs) when spatial Riemann–Liouville derivatives are employed, in addition to the *central* FLIs when spatial Riesz derivatives are employed.

The organization of the paper is as follows: in Section 2 we first present some preliminaries from fractional calculus. In Section 3, we define the general setting of the model problem in this study, and the field-variable fractional order derivatives of Riemann–Liouville and Riesz type are introduced. In Section 4, we construct three types of fractional Lagrange interpolants and investigate their properties. Next in Section 5, we obtain the associated differentiation matrices of variable-order, corresponding to each type of fractional derivatives. Subsequently, we investigate the performance of our fractional collocation spectral methods in the framework of linear and nonlinear variable-order FPDEs in Section 6. Moreover in this section, we propose a *penalty method* to impose inhomogeneous initial conditions. We finally conclude the paper with a summary and discussion in Section 7.

2. Preliminaries

We first provide some definitions from fractional calculus. Following [48], for a function $w(z) \in C^n[z_L, z_R]$, we denote by ${}_{z_L}^{\mathcal{D}}{}^\gamma w(z)$ the *left-sided* Riemann–Liouville fractional derivative of order γ , when $n - 1 \leq \gamma < n$, defined as

$${}_{z_L}^{\mathcal{D}}{}^\gamma w(z) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dz^n} \int_{z_L}^z \frac{w(s)}{(z-s)^{\gamma+1-n}} ds, \quad z \in [z_L, z_R], \quad (4)$$

where Γ represents the Euler gamma function, and as $\gamma \rightarrow n$, the *global* operator ${}_{z_L}^{\mathcal{D}}{}^\gamma \rightarrow d^n/dz^n$, recovering the *local* n -th order derivative with respect to z . We also denote by ${}_z^{\mathcal{D}}{}^\gamma w(z)$ the corresponding *right-sided* Riemann–Liouville fractional derivative of order $n - 1 \leq \gamma < n$, defined as

$${}_z^{\mathcal{D}}{}^\gamma w(z) = \frac{1}{\Gamma(n-\gamma)} (-1)^n \frac{d^n}{dz^n} \int_z^b \frac{w(s)}{(s-z)^{\gamma+1-n}} ds, \quad z \in [z_L, z_R]. \quad (5)$$

Similarly, as $\gamma \rightarrow n$, the right-sided fractional derivative tends to the standard n -th order local derivative. We also recall from [48] a useful property of the Riemann–Liouville fractional derivatives. Assume that $0 < p \leq 1$, $0 < q \leq 1$ and $z > z_L$, then

$${}_{z_L}^{\mathcal{D}}{}^{p+q} w(z) = ({}_{z_L}^{\mathcal{D}}{}^p)({}_{z_L}^{\mathcal{D}}{}^q) w(z) = ({}_{z_L}^{\mathcal{D}}{}^q)({}_{z_L}^{\mathcal{D}}{}^p) w(z), \quad (6)$$

when $w(z_L) = 0$, and

$${}_z^{\mathcal{D}}{}^{p+q} w(z) = ({}_z^{\mathcal{D}}{}^p)({}_z^{\mathcal{D}}{}^q) w(z) = ({}_z^{\mathcal{D}}{}^q)({}_z^{\mathcal{D}}{}^p) w(z), \quad (7)$$

when $w(z_R) = 0$. Moreover, for $k > -1$, we have

$${}_{z_L}^{\mathcal{D}}{}^\gamma (z + z_L)^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)} (z + z_L)^{k-\gamma} \quad (8)$$

and

$${}_z^{\mathcal{D}}{}^\gamma (z_R - z)^k = \frac{-\Gamma(k+1)}{\Gamma(k+1-\gamma)} (z_R - z)^{k-\gamma}, \quad (9)$$

if $\gamma \in (0, 1)$. In addition, let again $z = \frac{z_R + z_L}{2} + \frac{z_R - z_L}{2}\xi$ map $x \in [x_L, x_R]$ with $\xi \in [-1, 1]$, known as the *standard domain*, to $[z_L, z_R]$. Then,

$${}^{RL}_z \mathcal{D}_z^\gamma w(z) = \left(\frac{2}{z_R - z_L} \right)^\gamma {}^{RL}_{-1} \mathcal{D}_\xi^\gamma w(z(\xi)), \quad (10)$$

moreover,

$${}^{RL}_z \mathcal{D}_{z_R}^\gamma w(z) = \left(\frac{2}{z_R - z_L} \right)^\gamma {}^{RL}_\xi \mathcal{D}_1^\gamma w(z(\xi)), \quad (11)$$

which are useful in the derivation of the emerging differentiation matrices in our method.

The corresponding fractional derivatives of Caputo type, i.e., ${}^C_{z_R} \mathcal{D}_z^\gamma w(z)$ and ${}^C_z \mathcal{D}_{z_R}^\gamma w(z)$, are also defined by interchanging the order of differentiation and integration in (4) and (5) as

$${}^C_{z_L} \mathcal{D}_z^\gamma w(z) = \frac{1}{\Gamma(n - \gamma)} \int_{z_L}^z \frac{d^n w(s)}{ds^n} \frac{ds}{(z - s)^{\gamma+1-n}}, \quad z \in [z_L, z_R], \quad (12)$$

and

$${}^C_z \mathcal{D}_{z_R}^\gamma w(z) = \frac{(-1)^n}{\Gamma(n - \gamma)} \int_z^{z_R} \frac{d^n w(s)}{ds^n} \frac{ds}{(s - z)^{\gamma+1-n}}, \quad z \in [z_L, z_R], \quad (13)$$

which we will employ in FPDEs subject to non-homogeneous initial conditions. From the definition of the Caputo derivatives, the properties (8) and (9) become

$${}^C_{z_L} \mathcal{D}_z^\gamma (z + z_L)^k = \begin{cases} 0, & k < \gamma, \\ \frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)} (z + z_L)^{k-\gamma}, & 0 < \gamma \leq k, \end{cases} \quad (14)$$

and

$${}^C_z \mathcal{D}_{z_R}^\gamma (z_R - z)^k = \begin{cases} 0, & k < \gamma, \\ \frac{-\Gamma(k+1)}{\Gamma(k+1-\gamma)} (z_R - z)^{k-\gamma}, & 0 < \gamma \leq k. \end{cases} \quad (15)$$

We denote by $\frac{\partial^\gamma u}{\partial |z|^\gamma}$ the Riesz fractional derivative of order γ , when $n - 1 < \gamma < n$, defined in terms of left- and right-sided Riemann–Liouville fractional derivatives as

$$\frac{d^\gamma u}{d|z|^\gamma} = C_\gamma ({}^{RL}_{z_L} \mathcal{D}_z^\gamma w(z) + {}^{RL}_z \mathcal{D}_{z_R}^\gamma w(z)), \quad (16)$$

where $C_\gamma = -\frac{1}{2} \sec(\gamma\pi/2)$. It is easy to check that as $\gamma \rightarrow n$, asymptotically $\frac{d^\gamma(\cdot)}{d|z|^\gamma} \rightarrow d^n(\cdot)/dz^n$.

So far, we note that all the aforementioned fractional derivatives are of a *constant* order γ and are defined originally for the *univariate* function $w(z)$. In the sequel, we introduce a model problem, in which the fractional operators are defined through generalization of the given definitions to *multi-variate* functions, in which the corresponding fractional orders are *field* variables rather than constants.

3. Problem definition

Let ζ , σ and $v : \mathbb{R}^2 \rightarrow \mathbb{R}$ be *continuous* functions. We study the following nonlinear variable-order time- and space-fractional FPDE for all $(x, t) \in [a, b] \times [0, T]$ as

$${}^{RL}_0 \mathcal{D}_t^{\zeta(x,t)} u + g(u) * \mathcal{D}_x^{\sigma(x,t)} u = K * \mathcal{D}_x^{1+v(x,t)} u + f(u; x, t), \quad (17)$$

$$u(a, t) = u(b, t) = 0, \quad (18)$$

$$u(x, 0) = 0, \quad (19)$$

where $K > 0$ and $f(u; x, t)$ denotes the forcing- and/or reaction term. Moreover, the *temporal* order $\zeta(x, t) \in (0, 1)$, the *spatial advection* order $\sigma(x, t) \in (0, 1)$, and the *spatial diffusion* order $1 + v(x, t) \in (1, 2)$, i.e., $v(x, t) \in (0, 1)$. We shall discuss the regularity of the aforementioned field variable orders in Section 6. Here, ${}^{RL}_0 \mathcal{D}_t^{\zeta(x,t)} u$ represents the $\zeta(x, t)$ -th order left-sided *partial* time derivative of $u(x, t)$ of Riemann–Liouville type, defined as

$${}^{RL}_0 \mathcal{D}_t^{\zeta(x,t)} u(x, t) = \frac{1}{\Gamma[1 - \zeta(x, t)]} \frac{\partial}{\partial t} \int_0^t \frac{u(x, s) ds}{(t - s)^{\zeta(x,t)}}, \quad (20)$$

based on (4). We consider the space-fractional derivatives ${}^*\mathcal{D}_x$, associated with the advection and the diffusion terms in (17), to be of either (i) Riemann–Liouville or (ii) Riesz type, when homogeneous initial/boundary conditions are imposed. Alternatively, when inhomogeneous initial conditions are enforced, we consider the temporal derivative to be of Caputo type.

(I) ${}^*\mathcal{D}_x$ of Riemann–Liouville type. We define the corresponding advection partial fractional derivative ${}^*\mathcal{D}_x^{\sigma(x,t)}u$ of Riemann–Liouville type according to (4) and (5) and based on the direction of the transport velocity as

$${}^*\mathcal{D}_x^{\sigma(x,t)}u(x,t) \equiv \begin{cases} {}^{RL}\mathcal{D}_a^{\sigma(x,t)}u = \frac{1}{\Gamma[1-\sigma(x,t)]} \frac{\partial}{\partial x} \int_a^x \frac{u(r,t) dr}{(x-r)^{\sigma(x,t)}}, & g > 0, \\ {}^{RL}\mathcal{D}_b^{\sigma(x,t)}u = \frac{1}{\Gamma[1-\sigma(x,t)]} \left(-\frac{\partial}{\partial x}\right) \int_x^b \frac{u(r,t) dr}{(r-x)^{\sigma(x,t)}}, & g < 0. \end{cases} \quad (21)$$

In addition, we define the diffusion partial fractional derivative ${}^*\mathcal{D}_x^{1+\nu(x,t)}u$ of Riemann–Liouville type either as ${}^{RL}\mathcal{D}_a^{1+\nu(x,t)}u \equiv \frac{\partial}{\partial x} [{}^{RL}\mathcal{D}_a^{\nu(x,t)}u]$ or ${}^{RL}\mathcal{D}_b^{1+\nu(x,t)}u \equiv \frac{\partial}{\partial x} [{}^{RL}\mathcal{D}_b^{\nu(x,t)}u]$.

(II) ${}^*\mathcal{D}_x$ of Riesz type. We alternatively define the corresponding advection partial fractional derivative ${}^*\mathcal{D}_x^{\sigma(x,t)}u$ of Riesz type as

$${}^*\mathcal{D}_x^{\sigma(x,t)}u(x,t) \equiv \frac{\partial^{\sigma(x,t)}u}{\partial |x|^{\sigma(x,t)}} = C_{\sigma(x,t)} ({}^{RL}\mathcal{D}_a^{\sigma(x,t)}u + {}^{RL}\mathcal{D}_b^{\sigma(x,t)}u). \quad (22)$$

Moreover, from (6) and (7), we define the diffusion partial fractional derivative ${}^*\mathcal{D}_x^{1+\nu(x,t)}u$ to be of Riesz type as

$${}^*\mathcal{D}_x^{1+\nu(x,t)}u(x,t) = C_{1+\nu(x,t)} \frac{\partial}{\partial x} [{}^{RL}\mathcal{D}_a^{\nu(x,t)}u(x,t) + {}^{RL}\mathcal{D}_b^{\nu(x,t)}u(x,t)]. \quad (23)$$

In the following, we develop a fractional spectral collocation method for efficient solution of (17) based on the variable-order fractional differential operators defined here.

4. Fractional Lagrange interpolants (FLIs)

We define a set of *interpolation points* on which the corresponding Lagrange interpolants are obtained. Specifically, we employ a new family of left- and right-sided space–time *fractional Lagrange interpolants* (FLI) rather than utilizing the standard algebraic/trigonometric polynomial Lagrange basis functions. We shall demonstrate that such a construction leads to efficient computation of the corresponding differentiation matrices in addition to efficient approximations. Denoting by u_N an approximation of the solution in terms of such interpolators, we formulate our collocation method by requiring the residual of the problem i.e.,

$$R_N(x,t) = {}^{RL}\mathcal{D}_t^{\zeta(x,t)}u_N + g(u) {}^*\mathcal{D}_x^{\sigma(x,t)}u_N - K {}^*\mathcal{D}_x^{1+\nu(x,t)}u_N - f(u_N; x, t), \quad (24)$$

to vanish on the same set of grid points called *collocation points*.

Our fractional spectral collocation scheme is inspired by a new spectral theory developed for fractional Sturm–Liouville eigen-problems (FSLP) in [1]. The idea is to represent the solution to (17) in terms of new fractional (non-polynomial) basis functions, called *Jacobi polyfractonomials*, which are the eigenfunctions of the FSLP of first and second kind, explicitly given in (2) and (3). So, depending on the choice of the spatial fractional derivative ${}^*\mathcal{D}_x$ in (17), we introduce the corresponding fractional Lagrange interpolants.

4.1. Construction of FLI when ${}^*\mathcal{D}_x \equiv {}^{RL}\mathcal{D}_x$

Let $\alpha = \beta = -1$ in (2) and (3), which corresponds to the *regular* eigenfunctions of first and second kind, given as

$${}^{(1)}\mathcal{P}_n^\mu(\xi) = (1+\xi)^\mu P_{n-1}^{-\mu,\mu}(\xi), \quad \xi \in [-1, 1], \quad (25)$$

and

$${}^{(2)}\mathcal{P}_n^\mu(\xi) = (1-\xi)^\mu P_{n-1}^{\mu,-\mu}(\xi), \quad \xi \in [-1, 1], \quad (26)$$

where we recall that $\mu \in (0, 1)$. From the properties of the eigensolutions in [1], the left-sided fractional derivatives of (25) and (26) are given as

$${}^{RL}\mathcal{D}_\xi^\mu ({}^{(1)}\mathcal{P}_n^\mu(\xi)) = {}^{RL}\mathcal{D}_\xi^\mu ({}^{(2)}\mathcal{P}_n^\mu(\xi)) = \frac{\Gamma(n+\mu)}{\Gamma(n)} P_{n-1}(x), \quad (27)$$

where $P_{n-1}(x)$ denotes a Legendre polynomial of order $(n-1)$. Hence, in general, we can employ the univariate eigenfunctions (25) and (26) to construct the space–time *nodal* basis functions needed. However, here we alternatively construct suitable *nodal* basis functions based on the transport velocity and the corresponding choice of the spatial derivatives ${}^{RL}\mathcal{D}_x$ or ${}^{RL}\mathcal{D}_b$; see (21).

4.1.1. Left-sided FLIs when ${}^*\mathcal{D}_x \equiv {}^{RL}_a\mathcal{D}_x$

We seek solutions as a *nodal* expansion

$$u_N(x, t) = \sum_{m=1}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) L_m^\mu(x) \mathcal{T}_n^\tau(t), \quad (28)$$

where $L_m^\mu(x)$ represent the *left-sided* spatial FLIs and $\mathcal{T}_n^\tau(t)$ denote the *temporal* FLIs, which are defined on some interpolation points $a = x_1 < x_2 < \dots < x_{\mathcal{M}} = b$ and $0 = t_1 < t_2 < \dots < t_{\mathcal{N}} = T$ as

$$L_m^\mu(x) = \left(\frac{x - x_1}{x_m - x_1} \right)^\mu \prod_{\substack{k=1 \\ k \neq m}}^{\mathcal{M}} \left(\frac{x - x_k}{x_m - x_k} \right), \quad 2 \leq m \leq \mathcal{M} - 1, \quad (29)$$

all of fractional order $(\mathcal{M} + \mu - 1)$, and

$$\mathcal{T}_n^\tau(t) = \left(\frac{t}{t_n} \right)^\tau \prod_{\substack{q=1 \\ q \neq n}}^{\mathcal{N}} \left(\frac{t - t_q}{t_n - t_q} \right), \quad 2 \leq n \leq \mathcal{N}, \quad (30)$$

of fractional order $(\mathcal{N} + \tau - 1)$. Here, we call the superscript μ and τ as spatial and temporal *interpolation parameters*, respectively. We set these constant fractional parameters prior to solving (17) from the variable orders given, i.e., τ , σ , and ν . Moreover, we note that the fractional interpolants satisfy the Kronecker delta property, i.e., $H_m^\mu(x_k) = \delta_{km}$ and $L_n^\tau(t_q) = \delta_{qn}$, at interpolation points.

Because of the homogeneous Dirichlet boundary/initial condition(s) in (17), we only construct $L_m^\mu(x)$ for $m = 2, 3, \dots, \mathcal{M}$ when the maximum fractional order $1 + \nu \in (1, 2)$, where we set $u_N(x_1, t) = u_N(x_{\mathcal{M}}, t) = 0$. Moreover, when $\tau \in (0, 1)$, there are only $(\mathcal{N} - 1)$ fractional Lagrange interpolants $\mathcal{T}_n^\tau(t)$, $n = 2, 3, \dots, \mathcal{N}$, since we impose $u_N(x, t_1) = 0$.

4.1.2. Right-sided FLIs when ${}^*\mathcal{D}_x \equiv {}^{RL}_x\mathcal{D}_b$

In this case, we seek the solution as another *nodal* expansion given by

$$u_N(x, t) = \sum_{m=1}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) R_m^\mu(x) \mathcal{T}_n^\tau(t), \quad (31)$$

where $R_m^\mu(x)$ represent the *right-sided* spatial FLIs, defined on the interpolations points as

$$R_m^\mu(x) = \left(\frac{x_{\mathcal{M}} - x}{x_{\mathcal{M}} - x_m} \right)^\mu \prod_{\substack{k=1 \\ k \neq m}}^{\mathcal{M}} \left(\frac{x - x_k}{x_m - x_k} \right), \quad 2 \leq m \leq \mathcal{M} - 1, \quad (32)$$

which are all again of fractional order $(\mathcal{M} + \mu - 1)$.

4.2. Central FLIs when ${}^*\mathcal{D}_x \equiv \frac{\partial}{\partial|x|}$ of Riesz type

When the fractional derivatives in (17) are all of Riesz type, we seek the solution as

$$u_N(x, t) = \sum_{m=1}^{\mathcal{M}+1} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) h_m(x) \mathcal{T}_n^\tau(t), \quad (33)$$

where $h_m(x)$ represent the standard polynomial Lagrange interpolants, defined on the interpolations points as

$$h_m(x) = \prod_{\substack{k=1 \\ k \neq m}}^{\mathcal{M}+1} \left(\frac{x - x_k}{x_m - x_k} \right), \quad 2 \leq m \leq \mathcal{M}, \quad (34)$$

which are all of order $\mathcal{M}$. The polynomial choice of $h_m(x)$ is mainly due to the *co-existence* of the left- and right-sided fractional derivatives in the definition of the Riesz derivatives in (22) and (23). Hence, compared to the structure of the FLIs in (29) and (32), we call the FLIs presented in (33) *central* in-space interpolants. In fact, they can be viewed as the nodal representations of Legendre polynomials i.e., $P_{\mathcal{M}}(x)$, which simultaneously appear to be the regular eigenfunctions (25) and (26) asymptotically when $\mu \rightarrow 0$ setting $n = \mathcal{M} + 1$.

When the time-derivative is of Caputo type, which is a proper setting in which we can enforce the inhomogeneous initial conditions, we also use the standard Legendre bases in time, i.e.,

$$u_N(x, t) = \sum_{m=1}^{\mathcal{M}+1} \sum_{n=1}^{\mathcal{N}+1} u_N(x_m, t_n) h_m(x) h_n(t), \quad (35)$$

where

$$h_n(t) = \prod_{\substack{k=1 \\ k \neq n}}^{\mathcal{N}+1} \left(\frac{t - t_k}{t_m - t_k} \right), \quad 1 \leq n \leq \mathcal{N} + 1, \quad (36)$$

which allows us to properly penalize the initial condition to the problem and develop a stable scheme.

5. Fractional differentiation matrices

We derive the corresponding spatial and temporal fractional differentiation matrices assuming that the collocation and interpolation points coincide. We first choose the suitable expansion among those given in (28), (31), or (33) based on the choice of the spatial fractional derivative ${}^*\mathcal{D}_x$. Then, we derive the corresponding differentiation matrices.

5.1. ${}^*\mathcal{D}_x$ of left-sided Riemann–Liouville type

Whether left- or right-sided Riemann–Liouville derivatives are employed in (17), we obtain the corresponding to left- or right-sided fractional differentiation matrices of order $\sigma(x, t)$ and $1 + \nu(x, t)$.

Theorem 5.1. Let $\sigma = \sigma(x, t) \in C([a, b] \times [0, T])$ and consider the affine mapping that maps $\xi \in [-1, 1]$ to $x \in [a, b]$. Then, when ${}^*\mathcal{D}_x \equiv {}^{RL}_a \mathcal{D}_x$ in (17), the left-sided spatial differentiation matrix ${}^{RL}\mathbf{D}_L^\sigma$ of Riemann–Liouville type, corresponding to the nodal FLI expansion (28), is a three-dimensional matrix whose entries are given by

$$\{{}^{RL}\mathbf{D}_L^\sigma\}_{ikm} = \left(\frac{2}{b-a} \right)^{\sigma(x_i, t_k)} A_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L F_j^{L, \sigma}(x_i, t_k), \quad (37)$$

where $i, m = 2, 3, \dots, \mathcal{M}, k = 2, 3, \dots, \mathcal{N}$, also $F_j^{L, \sigma}(x(\xi), t)$ is explicitly given as

$$F_j^{L, \sigma}(x(\xi), t) = \sum_{q=0}^{j-1} b_{jq}^\mu (1 + \xi)^{q+\mu-\sigma}, \quad (38)$$

in which $A_m = \left(\frac{b-a}{2x_m-2a} \right)^\mu$, and finally β_{mj}^L and b_{jq}^μ are the corresponding expansion coefficients, given a priori by (A.1) and (A.7).

Proof. Given in Appendix A. $\square$

Remark 5.2. In standard collocation methods applied to constant/integer-order operators, the corresponding differentiation matrices are two-dimensional. The extra dimension appearing in the left-sided differentiation matrix ${}^{RL}\mathbf{D}_L^\sigma$ is due to the field-variable $\sigma(x, t)$, which makes the corresponding fractional differential operator vary across the computational domain. Consequently, by collocating the fractional order on each collocation point (x_i, t_k) , we must compute all the entries associated with the whole set of the spatial points indexed by “ m ”. Therefore, it naturally renders the corresponding differentiation matrix three-dimensional. Interestingly, when $\sigma = \sigma(x)$, we reduce the dimension of ${}^{RL}\mathbf{D}_L^\sigma$ by one, and we obtain the entries of the two-dimensional differentiation matrix as

$$\{{}^{RL}\mathbf{D}_L^\sigma\}_{im} = \left(\frac{2}{b-a} \right)^{\sigma(x_i)} A_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L F_j^{L, \sigma}(x_i). \quad (39)$$

Moreover, when σ is constant, the differentiation matrix in (39) is further reduced to

$$\{{}^{RL}\mathbf{D}_L^\sigma\}_{im} = \left(\frac{2}{b-a} \right)^\sigma A_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \frac{\Gamma(j+\sigma)}{\Gamma(j)} P_{j-1}(\xi_i), \quad (40)$$

previously given in [46], where $x_i = \frac{a+b}{2} + \frac{b-a}{2} \xi_i$.

The next theorem provides the corresponding left-sided differentiation matrix for the diffusion term in (17) when ${}^*\mathcal{D}_x^{1+\nu} \equiv {}^{RL}_a \mathcal{D}_x^{1+\nu}$.

Theorem 5.3. Let $v = v(x, t) \in C([a, b] \times [0, T])$ and consider the affine mapping the maps $\xi \in [-1, 1]$ to $x \in [a, b]$. Then, the left-sided spatial differentiation matrix ${}^{RL}\mathbf{D}_L^{1+v}$ of Riemann–Liouville type, corresponding to the nodal FLI expansion (28) when ${}^*\mathcal{D}_x \equiv {}^{RL}\mathcal{D}_x$ in (17), is a three-dimensional matrix whose entries are given by

$$\{{}^{RL}\mathbf{D}_L^{1+v}\}_{ikm} = \left(\frac{2}{b-a}\right)^{1+v(x_i, t_k)} A_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \mathcal{F}_j^{L,v}(x_i, t_k), \quad (41)$$

in which $A_m = (\frac{b-a}{2x_m-2a})^\mu$, $i, m = 2, 3, \dots, \mathcal{M}$, $k = 2, 3, \dots, \mathcal{N}$, and $\mathcal{F}_j^{L,v}(x(\xi), t)$ is explicitly given by

$$\begin{aligned} \mathcal{F}_j^{L,v}(x(\xi), t) &= \mathbb{I}_{\{j \geq 1\}} \sum_{q=0}^{j-1} \mathbb{B}_{jq}^\mu \cdot (1+\xi)^{q+\mu-1-v(x_i, t_k)} \\ &\quad + \mathbb{I}_{\{j \geq 2\}} \sum_{q=0}^{j-2} B_{jq}^\mu \cdot (1+\xi)^{q+\mu-v(x_i, t_k)}, \end{aligned} \quad (42)$$

where $\mathbb{B}_{jq}^\mu$ and B_{jq}^μ are the corresponding expansion coefficients, given a priori by (B.4) and (B.5).

Proof. Given in Appendix B. $\square$

In analogy with Remark 5.2, we also appreciate the appearance of the extra dimension in the right-sided differentiation matrix ${}^{RL}\mathbf{D}_L^{1+v}$. Similarly, when $v = v(x)$, we reduce the dimension of ${}^{RL}\mathbf{D}_L^{1+v}$ by one and obtain the entries of the two-dimensional differentiation matrix as

$$\{{}^{RL}\mathbf{D}_L^{1+v}\}_{im} = \left(\frac{2}{b-a}\right)^{1+v(x_i)} A_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \mathcal{F}_j^{L,v}(x_i). \quad (43)$$

Remark 5.4. We note that the coefficients β_{mj}^L , shown in (A.1), are obtained only once and are utilized as many times as needed to construct ${}^{RL}\mathbf{D}_L^\sigma$ and ${}^{RL}\mathbf{D}_L^{1+v}$ for any order σ , $v \in (0, 1)$.

5.2. ${}^*\mathcal{D}_x$ of right-sided Riemann–Liouville type

Following similar steps, presented in Section 5.1, we now construct the corresponding right-sided advection differentiation matrix of order $\sigma(x, t)$ and the right-sided diffusion differentiation matrix of order $1 + v(x, t)$ when ${}^*\mathcal{D}_x \equiv {}^{RL}_x\mathcal{D}_b$ in the sequel.

Theorem 5.5. Let $\sigma = \sigma(x, t) \in C([a, b] \times [0, T])$ and consider the affine mapping that maps $\xi \in [-1, 1]$ to $x \in [a, b]$. Then, when ${}^*\mathcal{D}_x \equiv {}^{RL}_x\mathcal{D}_b$ in (17), the right-sided spatial differentiation matrix ${}^{RL}\mathbf{D}_R^\sigma$ of Riemann–Liouville type, corresponding to the nodal FLI expansion (31), is a three-dimensional matrix whose entries are given by

$$\{{}^{RL}\mathbf{D}_R^\sigma\}_{ikm} = \left(\frac{2}{b-a}\right)^{\sigma(x_i, t_k)} \mathcal{A}_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^R F_j^{R,\sigma}(x_i, t_k) \quad (44)$$

in which $i, m = 2, 3, \dots, \mathcal{M}$, $k = 2, 3, \dots, \mathcal{N}$, also $F_j^{R,\sigma}(x(\xi), t)$ is explicitly given as

$$F_j^{R,\sigma}(x(\xi), t) = \sum_{q=0}^{j-1} c_{jq}^\mu (1-\xi)^{q+\mu-\sigma}, \quad (45)$$

in which $\mathcal{A}_m = 1/(\xi_{\mathcal{M}} - \xi_m)^\mu$, and finally β_{mj}^R and c_{jq}^μ are the corresponding expansion coefficients, given a priori by (C.1) and (C.6).

Proof. Given in Appendix C. $\square$

The next theorem provides the corresponding right-sided differentiation matrix for the diffusion term in (17) when ${}^*\mathcal{D}_x^{1+v} \equiv {}^{RL}_x\mathcal{D}_b^{1+v(x,t)}$.

Theorem 5.6. Let $v = v(x, t) \in C([a, b] \times [0, T])$ and consider the affine mapping that maps $\xi \in [-1, 1]$ to $x \in [a, b]$. Then, the right-sided spatial differentiation matrix ${}^{RL}\mathbf{D}_R^{1+v}$ of Riemann–Liouville type, corresponding to the nodal FLI expansion (31) when ${}^*\mathcal{D}_x \equiv {}^{RL}_x\mathcal{D}_b$ in (17), is a three-dimensional matrix whose entries are given by

$$\{ {}^{RL}\mathbf{D}_R^{1+\nu} \}_{ikm} = \left(\frac{2}{b-a} \right)^{1+\nu(x_i, t_k)} \mathcal{A}_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^R \mathcal{F}_j^{R, \nu}(x_i, t_k), \quad (46)$$

where $i, m = 2, 3, \dots, \mathcal{M}, k = 2, 3, \dots, \mathcal{N}$, and $\mathcal{F}_j^{R, \nu}(x(\xi), t)$ is explicitly given by

$$\begin{aligned} \mathcal{F}_j^{R, \nu}(x(\xi), t) = & \mathbb{I}_{\{j \geq 1\}} \sum_{q=0}^{j-1} \mathbb{C}_{jq}^{\mu} \cdot (1-\xi)^{q+\mu-1-\nu(x_i, t_k)} \\ & + \mathbb{I}_{\{j \geq 2\}} \sum_{q=0}^{j-2} \mathbb{C}_{jq}^{\mu} \cdot (1-\xi)^{q+\mu-\nu(x_i, t_k)}, \end{aligned} \quad (47)$$

in which $\mathbb{C}_{jq}^{\mu}$ and $\mathbb{C}_{jq}^{\mu}$ are the corresponding expansion coefficients, given a priori by (D.3) and (D.2).

Proof. Given in Appendix D. $\square$

5.3. ${}^*\mathcal{D}_x$ of Riesz type

The following lemma is useful in the derivation and construction of the Riesz spatial differentiation matrices.

Lemma 5.7. (See [49].) For $\mu > 0$, $\alpha > -1$, $\beta > -1$, and $\forall \xi \in [-1, 1]$

$$(1+\xi)^{\beta+\mu} \frac{P_n^{\alpha-\mu, \beta+\mu}(\xi)}{P_n^{\alpha-\mu, \beta+\mu}(-1)} = \frac{\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)P_n^{\alpha, \beta}(-1)} \int_{-1}^{\xi} \frac{(1+s)^{\beta} P_n^{\alpha, \beta}(s)}{(\xi-s)^{1-\mu}} ds, \quad (48)$$

and

$$(1-\xi)^{\alpha+\mu} \frac{P_n^{\alpha+\mu, \beta-\mu}(\xi)}{P_n^{\alpha+\mu, \beta-\mu}(+1)} = \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)P_n^{\alpha, \beta}(+1)} \int_{\xi}^1 \frac{(1-s)^{\alpha} P_n^{\alpha, \beta}(s)}{(s-\xi)^{1-\mu}} ds. \quad (49)$$

By the definition of the left-sided Riemann–Liouville integral ${}^{RL}_{-1}\mathcal{I}_{\xi}^{\mu}$ and evaluating the special end-values $P_n^{\alpha-\mu, \beta+\mu}(-1)$ and $P_n^{\alpha, \beta}(-1)$, we can re-write (48) as

$${}^{RL}_{-1}\mathcal{I}_{\xi}^{\mu} \{ (1+\xi)^{\beta} P_n^{\alpha, \beta}(\xi) \} = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+\mu+1)} (1+\xi)^{\beta+\mu} P_n^{\alpha-\mu, \beta+\mu}(\xi).$$

Now, by taking the fractional derivative ${}^{RL}_{-1}\mathcal{D}_{\xi}^{\mu}$ on the when $\beta = -\mu$ and $\alpha = \mu$ we obtain

$${}^{RL}_{-1}\mathcal{D}_{\xi}^{\mu} \{ P_n(\xi) \} = \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)} (1+\xi)^{-\mu} P_n^{\mu, -\mu}(\xi). \quad (50)$$

Similarly, by the definition of the right-sided Riemann–Liouville integral ${}^{RL}_{\xi}\mathcal{I}_1^{\mu}$ and evaluating the special end-values $P_n^{\alpha-\mu, \beta+\mu}(+1)$ and $P_n^{\alpha, \beta}(+1)$, we can re-write (49) as

$${}^{RL}_{\xi}\mathcal{I}_1^{\mu} \{ (1-\xi)^{\alpha} P_n^{\alpha, \beta}(\xi) \} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\mu+1)} (1-\xi)^{\alpha+\mu} P_n^{\alpha+\mu, \beta-\mu}(\xi).$$

In a similar fashion, by taking the fractional derivative ${}^{RL}_{\xi}\mathcal{D}_{-1}^{\mu}$ on the both sides when $\alpha = -\mu$ and $\beta = \mu$ we obtain

$${}^{RL}_{\xi}\mathcal{D}_1^{\mu} \{ P_n(\xi) \} = \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)} (1-\xi)^{-\mu} P_n^{-\mu, \mu}(\xi). \quad (51)$$

Theorem 5.8. Let $\sigma = \sigma(x, t) \in C([a, b] \times [0, T])$ and consider the affine mapping that maps $\xi \in [-1, 1]$ to $x \in [a, b]$. Then, when ${}^*\mathcal{D}_x \equiv \partial \sigma(x, t) / \partial |x|^{\sigma(x, t)}$ in (17), the right-sided spatial differentiation matrix $\mathbf{D}_{\text{Riesz}}^{\sigma}$ of Riesz type, corresponding to the nodal FLI expansion (33), is a three-dimensional matrix whose entries are given by

$$\{ \mathbf{D}_{\text{Riesz}}^{\sigma} \}_{ikm} = \left[\left(\frac{2}{b-a} \right)^{\sigma(x, t)} C_{\sigma(x, t)} \right]_{(x_i, t_k)} \sum_{j=1}^{\mathcal{M}} \tilde{\beta}_{mj} \mathcal{Z}_j^{\sigma}(x_k, t_k), \quad (52)$$

in which $i, m = 2, 3, \dots, \mathcal{M}, k = 2, 3, \dots, \mathcal{N}, \xi = 2 \frac{x-a}{b-a}$, and $\tilde{\beta}_{mj}$ are the corresponding expansion coefficients, given a priori by (E.1). Moreover, $\mathcal{Z}_j^\sigma(x, t)$ is explicitly given as

$$\mathcal{Z}_j^\sigma(x, t) = \frac{\Gamma(j+1)}{\Gamma(j-\sigma(x_i, t_k) + 1)} \left[(1+\xi)^{-\sigma(x, t)} P_j^{\sigma, -\sigma}(\xi) + (1-\xi)^{-\sigma(x, t)} P_j^{-\sigma, \sigma}(\xi) \right].$$

Proof. Given in Appendix E. $\square$

Theorem 5.9. Let $v = v(x, t) \in C([a, b] \times [0, T])$ and consider the affine mapping that maps $\xi \in [-1, 1]$ to $x \in [a, b]$. Then, when ${}^*\mathcal{D}_x \equiv \partial^{1+v(x, t)} u / \partial |x|^{1+v(x, t)}$ in (17), the right-sided spatial differentiation matrix $\mathbf{D}_{\text{Riesz}}^{1+v}$ of Riesz type, corresponding to the nodal FLI expansion (33), is a three-dimensional matrix whose entries are given by

$$\{\mathbf{D}_{\text{Riesz}}^{1+v(x, t)}\}_{ikm} = \left[\left(\frac{2}{b-a} \right)^{1+v} c_{1+v} \right]_{(x_i, t_k)} \sum_{j=1}^{\mathcal{M}} \tilde{\beta}_{mj} \mathcal{W}_j^\sigma(x, t) \quad (53)$$

in which $i, m = 2, 3, \dots, \mathcal{M}, k = 2, 3, \dots, \mathcal{N}, \xi = 2 \frac{x-a}{b-a}$, and $\mathcal{W}_j^\sigma(x, t)$ is explicitly given as

$$\mathcal{W}_j^\nu = \left(\frac{j+1}{2} \right) \sum_{q=\lceil \nu \rceil}^{j-1} c_j \frac{(-\frac{1}{2})^q \Gamma(q+1)}{\Gamma(q+1-\nu(x, t))} \left[(-1)^{j-1} (1+\xi)^{q-\nu(x, t)} - (1-\xi)^{q-\nu(x, t)} \right].$$

Proof. Given in Appendix F. $\square$

5.4. Temporal differentiation matrix ${}^{RL}\mathbf{D}_t^\tau$

We derive the temporal differentiation matrix ${}^{RL}\mathbf{D}_t^\zeta$ by taking ${}^{RL}_0\mathcal{D}_t^{\zeta(x, t)} u_N(x, t)$ and evaluating it the collocation points considering the fact that the spatial nodal bases $L_m^\mu(x)$, $R_m^\mu(x)$, and $h_m(x)$ all satisfy the Kronecker delta property at the collocation points. Hence, in the derivation of ${}^{RL}\mathbf{D}_t^\zeta$ any of the expansions (28), (31), or (33) can be used. Following similar steps in Section 5.1, we obtain the temporal differentiation matrix corresponding to the following two cases:

Case I-A (Constant $\zeta = \tau \in (0, 1)$). We map the interval $t \in [0, T]$ to the standard domain $\xi \in [-1, 1]$ as usual, and use the property (27) to obtain

$${}^{RL}_0\mathcal{D}_t^\zeta u_N(x, t)|_{(x_i, t_k)} = \sum_{n=2}^{\mathcal{N}} \{ {}^{RL}\mathbf{D}_t^\zeta \}_{kn} u_N(x_i, t_n),$$

where $\{ {}^{RL}\mathbf{D}_t^\zeta \}_{im}$ are the entries of the $(\mathcal{N}-1) \times (\mathcal{N}-1)$ left-sided temporal differentiation matrix ${}^{RL}\mathbf{D}_t^\zeta$ of Riemann–Liouville since, given by

$$\{ {}^{RL}\mathbf{D}_t^\zeta \}_{kn} = \left(\frac{2}{T} \right)^\zeta \eta_n \sum_{j=1}^{\mathcal{N}} \beta_{nj}^L \frac{\Gamma(j+\zeta)}{\Gamma(j)} P_{j-1}(\xi_k), \quad (54)$$

where $\eta_n = \frac{T}{2t_n}$ and $\xi_k = \frac{2t_k}{T} - 1$.

Case I-B (The general $\zeta(x, t) \in (0, 1)$). We obtain the corresponding temporal differentiation matrix of variable order $\zeta(x, t) \in (0, 1)$, by evaluating ${}^{RL}_0\mathcal{D}_x^{\zeta(x, t)} u_N(x, t)$ at the collocation points (x_i, t_k) also by $L_m^\mu(x_i) = \delta_{im}$, we obtain

$$\begin{aligned} {}^{RL}_0\mathcal{D}_t^{\zeta(x, t)} u_N|_{(x_i, t_k)} &= \left(\frac{2}{T} \right)^{\zeta(x_i, t_k)} \sum_{n=2}^{\mathcal{N}} u_N(x_m, t_k) \eta_n \sum_{j=1}^{\mathcal{N}} \beta_{nj}^L F_j^{L, \tau}(x_i, t_k) \\ &= \sum_{n=2}^{\mathcal{N}} \{ {}^{RL}\mathbf{D}_t^\zeta \}_{kni} u_N(x_i, t_n), \end{aligned}$$

where $\{ {}^{RL}\mathbf{D}_t^\zeta \}_{kni}$ are the entries of the $(\mathcal{N}-1) \times (\mathcal{M}-1) \times (\mathcal{N}-1)$ left-sided temporal fractional differentiation matrix ${}^{RL}\mathbf{D}_t^\zeta$ of Riemann–Liouville sense, computed as

$$\{ {}^{RL}\mathbf{D}_t^\zeta \}_{ikn} = \left(\frac{2}{T} \right)^{\zeta(x_i, t_k)} \eta_n \sum_{j=1}^{\mathcal{N}} \beta_{nj}^L F_j^{L, \tau}(x_i, t_k). \quad (55)$$

We note that β_{nj}^L are the corresponding coefficients in the following expansion that are obtained once as

$$\mathcal{G}_n(\xi) = \sum_{j=1}^{\mathcal{N}} \beta_{nj}^L P_{j-1}^{-\tau, \tau}(\xi), \quad (56)$$

similar to what we showed in (A.1).

5.5. Temporal differentiation matrix ${}^C\mathbf{D}_t^\zeta$

We recall that when the time-derivative is of Caputo type we employ the expansion (35).

Theorem 5.10. Let $\zeta = \zeta(x, t) \in C([a, b] \times [0, T])$ and consider the affine mapping that maps $\eta \in [-1, 1]$ to $t \in [0, T]$. Then, when ${}^*\mathcal{D}_t \equiv {}^C_0\mathcal{D}_t$ in (17), the left-sided temporal differentiation matrix ${}^C\mathbf{D}^\zeta$ of Caputo type, corresponding to the nodal FLI expansion (35), is a three-dimensional matrix whose entries are given by

$$\{{}^C\mathbf{D}^\zeta\}_{ikm} = \left[\left(\frac{2}{T} \right)^{\zeta(x,t)} \right]_{(x_i, t_k)} \sum_{j=1}^{\mathcal{N}} \tilde{\beta}_{mj} \mathcal{Z}_j^\zeta(x_k, t_k), \quad (57)$$

when the solution to (17) is assumed to be continuous in time, in which $i, m = 2, 3, \dots, \mathcal{M}, k = 1, 2, 3, \dots, \mathcal{N} + 1, \eta = \frac{2t}{T} - 1$, and $\tilde{\beta}_{mj}$ are the corresponding (temporal) expansion coefficients, given a priori similarly as in (E.1). Moreover, $\mathcal{Z}_j^\zeta(x, t)$ is explicitly given as

$$\mathcal{Z}_j^\zeta(x, t) = \frac{\Gamma(j+1)}{\Gamma(j - \zeta(x_i, t_k) + 1)} [(1 + \eta)^{-\zeta(x,t)} P_j^{\zeta, -\zeta}(\eta)].$$

Proof. See Appendix E. $\square$

6. Numerical tests

After the construction of the variable-order differentiation matrices of Riemann–Liouville and Riesz type, we now solve a number of FPDEs to investigate the performance of our schemes. We divide this section into two main parts. In the first part, we implement our variable-order collocation method in solving *linear* FPDEs such as time- and space-fractional *advection* equation, *diffusion*, and *advection–diffusion* problems. In the second part, we deal with *nonlinear* FPDEs of field-variable order, namely the space-fractional nonlinear Burgers equation.

In all the numerical tests, we set collocation and interpolation points to be identical. Here, we adopt the choice of collocation points in [46], where we showed that the *fractional extrema* of ${}^{(1)}\mathcal{P}_n^\mu(\xi)$ and ${}^{(2)}\mathcal{P}_n^\mu(\xi)$, i.e., the zeros of Legendre polynomials, are the best collocation points leading to the fastest rate of convergence. To demonstrate the accuracy of our methods, we adopt the L^∞ norm, normalized by the essential norm of the exact solution in each case.

6.1. Linear FPDEs with ${}^*\mathcal{D}_x \equiv {}^{RL}\mathcal{D}_x$

We first consider two examples of linear FPDEs, namely advection and advection–diffusion problems, in which the spatial fractional derivatives are all either of left- or right-sided Riemann–Liouville type.

Example 6.1. Fractional advection equation:

$${}^{RL}_0\mathcal{D}_t^{\zeta(x,t)} u + \theta {}^{RL}\mathcal{D}_x^{\sigma(x,t)} u = f(x, t) \quad (x, t) \in [-1, 1] \times [0, 2], \quad (58)$$

$$u(x, 0) = 0. \quad (59)$$

where we replace $g(u)$ in (17) with the constant advection velocity θ . Here, ${}^{RL}\mathcal{D}_x^\sigma \equiv {}^{RL}_{-1}\mathcal{D}_x^\sigma$ when $\theta > 0$, hence we set the boundary condition $u(-1, t) = 0$, and ${}^{RL}\mathcal{D}_x^\sigma \equiv {}^{RL}_x\mathcal{D}_1^\sigma$ when $\theta < 0$, and therefore we set $u(1, t) = 0$. Moreover in (58), $\zeta, \sigma: \mathbb{R}^2 \rightarrow \mathbb{R}$ are *continuous* functions. In fact, we deduce the continuity requirement in ζ and σ from the definition of the corresponding temporal and spatial differentiation matrices in (55) and (A.8), implying that each entry is assumed to be continuous from above and below. Let $\theta = 1$ and seek the solution of the form (28), where we set the interpolation parameters τ and μ to be the *mean-value* of $\zeta(x, t)$ and $\sigma(x, t)$ given by

$$\tau = \frac{1}{|\Omega|} \int_{\Omega} \zeta d\Omega, \quad (60)$$

and

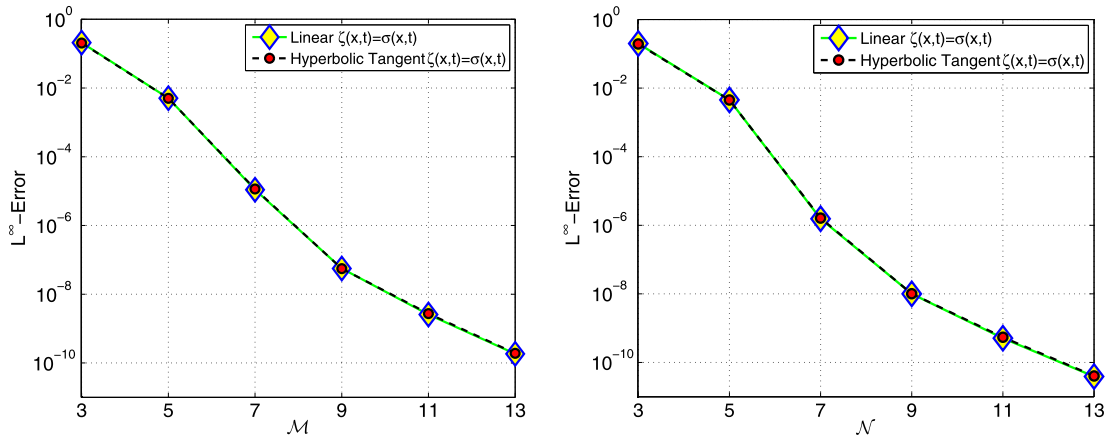


Fig. 2. Time- and space-fractional linear advection problem with Riemann–Liouville spatial operators: (left) spatial p -refinement, and (right) temporal p -refinement. The exact solution is the fractional function $u^{ext}(x, t) = (1 + x)^{6+9/17}t^{6+2/3}$, where $(x, t) \in [-1, 1] \times [0, 2]$; moreover, the temporal and spatial fractional orders are taken as the following field-variable functions, denoted as *linear* $\zeta(x, t) = \sigma(x, t) = (\frac{5+4x}{10})(\frac{1+4t}{10})$, and *hyperbolic tangent* $\zeta(x, t) = \sigma(x, t) = [1 + \tanh(x)][1 + \tanh(t-1)]/4$.

$$\mu = \frac{1}{|\Omega|} \int_{\Omega} \sigma d\Omega, \quad (61)$$

receptively. This is analogous to the algebraic mean-value interpolation parameters in multi-term FODEs/FPDEs, studied in [46]. We have investigated that such mean-values also lead to efficient approximation of fractional operators of variable-orders. Next, by substituting $u_N(x, t)$ in (58), we require the residual

$$R_N(x, t) = {}^{RL}_0 D_t^{\zeta(x,t)} u_N + {}^{RL}_{-1} D_x^{\sigma(x,t)} u_N - f(x, t), \quad (62)$$

to vanish at the collocation points (x_i, t_k) , which leads to the following linear system:

$${}^{RL}S_A \vec{u} = \vec{b}, \quad (63)$$

where ${}^{RL}S_A$ is an $(\mathcal{M} - 1)(\mathcal{N} - 1) \times (\mathcal{M} - 1)(\mathcal{N} - 1)$ matrix, whose entries are given by

$${}^{RL}S_A\{I_{ik}, J_{mn}\} = \{{}^{RL}\mathbf{D}_t^{\zeta}\}_{ikn} \delta_{im} + \{{}^{RL}\mathbf{D}_x^{\sigma}\}_{ikm} \delta_{kn}, \quad (64)$$

where $I_{ik} = (i - 2)(\mathcal{N} - 1) + k - 1$, $J_{mn} = (m - 2)(\mathcal{N} - 1) + n - 1$, also $\{{}^{RL}\mathbf{D}_t^{\zeta}\}_{ikn}$ and $\{{}^{RL}\mathbf{D}_x^{\sigma}\}_{ikm}$ are given in (55) and (A.8), respectively. Moreover, $\vec{b}$ represents the load vector whose components are obtained as

$$\vec{b}\{I_{ik}\} = f(x_i, t_k), \quad (65)$$

and finally, the vector of unknown coefficients is defined as

$$\vec{u}\{J_{mn}\} = u(x_m, t_n), \quad (66)$$

where $i, m = 2, 3, \dots, \mathcal{M}$ and $k, n = 2, 3, \dots, \mathcal{N}$.

In Fig. 2, we plot the exponential-like decay of L^∞ -norm of the error in corresponding spatial and temporal p -refinement. We set the exact solution to be the following fractional function $u^{ext}(x, t) = (1 + x)^{6+9/17}t^{6+2/3}$. Moreover, the temporal and spatial fractional orders are taken as the following field-variable functions, denoted as *linear* $\zeta(x, t) = \sigma(x, t) = (\frac{5+4x}{10})(\frac{1+4t}{10})$, and *hyperbolic tangent* $\zeta(x, t) = \sigma(x, t) = [1 + \tanh(x)][1 + \tanh(t-1)]/4$. We obtain identical results when we choose $\theta = -1$, i.e., the wind blows from right to left, and therefore right-sided Riemann–Liouville fractional derivatives are employed, considering $u^{ext}(x, t) = (1 - x)^{6+9/17}t^{6+2/3}$.

Example 6.2. Fractional advection–diffusion equation:

$${}^{RL}_0 D_t^{\zeta(x,t)} u + \theta {}^{RL}D_x^{\sigma(x,t)} u = {}^{RL}D_x^{1+\nu(t)} u + f(x, t) \quad (x, t) \in [-1, 1] \times [0, 2], \quad (67)$$

$$u(\pm 1, t) = 0 \quad (68)$$

$$u(x, 0) = 0. \quad (69)$$

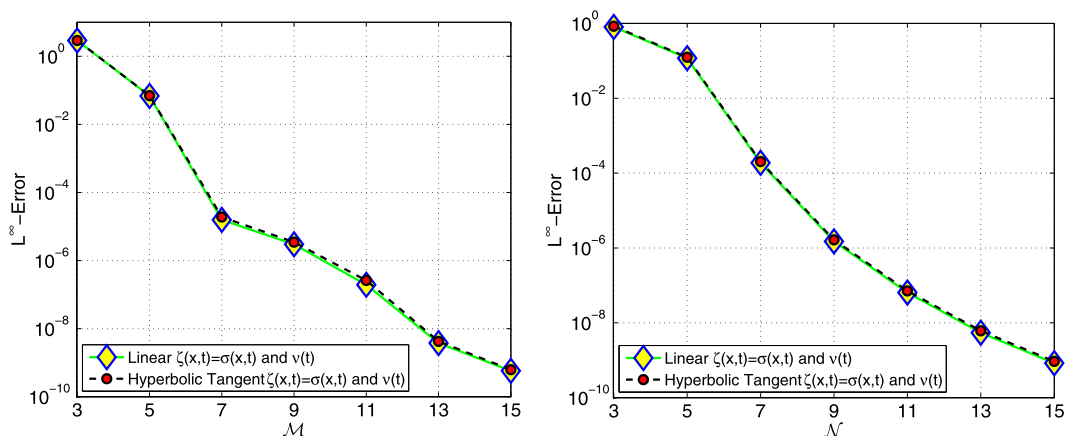


Fig. 3. Time- and space-fractional linear advection–diffusion equation with Riemann–Liouville spatial operators: (left) spatial p -refinement, and (right) temporal p -refinement. The exact solution is given by the fractional $u^{ext}(x, t) = (1 + x)^{6+9/17}t^{6+2/3}$, where $(x, t) \in [-1, 1] \times [0, 2]$; moreover, the temporal and spatial fractional orders are respectively taken as the following field-variable functions, denoted as linear $\zeta(x, t) = \sigma(x, t) = (\frac{5+4x}{10})(\frac{1+4t}{10})$ and $1 + \nu(t) = 1 + \frac{1+4t}{10}$, also hyperbolic tangent $\zeta(x, t) = \sigma(x, t) = [1 + \tanh(x)][1 + \tanh(t - 1)]/4$ and $1 + \nu(t) = [3 + \tanh(t - 1)]/2$.

Here, ${}^{RL}\mathcal{D}_x^\sigma \equiv {}^{RL}\mathcal{D}_x^\sigma$ and ${}^{RL}\mathcal{D}_x^{1+\nu} u \equiv {}^{RL}\mathcal{D}_x^{1+\nu} u$ if $\theta > 0$, i.e., when the wind blows from left to right. Alternatively, we set ${}^{RL}\mathcal{D}_x^\sigma \equiv {}^{RL}\mathcal{D}_x^\sigma$ and ${}^{RL}\mathcal{D}_x^{1+\nu} u \equiv {}^{RL}\mathcal{D}_x^{1+\nu} u$ when $\theta < 0$. We note that in (67), $\zeta : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions. In addition, $\nu : \mathbb{R} \rightarrow \mathbb{R}$ suffices to be *continuous* in order for the entries of the differentiation matrices to be well-defined. We first set $\theta = 1$ and seek the solution of the form (31), where we set the interpolation parameter μ to be the *mean-value* of $\sigma(x, t)$ given in (61). In a similar fashion, we can obtain the total average temporal interpolation parameter τ by including both $\zeta(x, t)$ and $\nu(t)$ into the averaging.

Having set the interpolation parameters, we substitute $u_N(x, t)$ in (67), and obtain the corresponding linear system by requiring the residual

$$R_N(x, t) = {}^{RL}\mathcal{D}_t^{\zeta(x, t)} u_N + {}^{RL}\mathcal{D}_x^{\sigma(x, t)} u_N + {}^{RL}\mathcal{D}_x^{1+\nu(t)} u_N - f(x, t), \quad (70)$$

to vanish at (x_i, t_k) . It then leads to

$${}^{RL}\mathcal{S}_{AD} \vec{u} = \vec{b}, \quad (71)$$

where ${}^{RL}\mathcal{S}_{AD}$ is an $(\mathcal{M} - 2)(\mathcal{N} - 1) \times (\mathcal{M} - 2)(\mathcal{N} - 1)$ matrix, whose entries are given by

$${}^{RL}\mathcal{S}_{AD}\{I_{ik}, J_{mn}\} = \{{}^{RL}\mathbf{D}_L^\zeta\}_{ikn}\delta_{im} + (\{{}^{RL}\mathbf{D}_L^\sigma\}_{ikm} - \{{}^{RL}\mathbf{D}_L^{1+\nu}\}_{ikm})\delta_{kn}, \quad (72)$$

in which $I_{ik} = (i - 2)(\mathcal{N} - 1) + k - 1$, $J_{mn} = (m - 2)(\mathcal{N} - 1) + n - 1$, also $\{{}^{RL}\mathbf{D}_L^{1+\nu}\}_{ikm}$ is given in (B.2), where $i, m = 2, 3, \dots, \mathcal{M} - 1$ and $k, n = 2, 3, \dots, \mathcal{N}$.

In order to examine the temporal and spatial accuracy of our schemes for this model problem, we plot the L^∞ -error versus expansion order in each case in Fig. 3. We set the exact solution again as $u^{ext}(x, t) = (1 + x)^{6+9/17}t^{6+2/3}$, where the temporal and spatial fractional orders are taken as the field-variable functions, denoted as linear and hyperbolic tangent. Once again, we observe the exponential-like decays of error. We found identical convergence results in the case $\theta = -1$, where the corresponding spatial derivatives are of right-sided Riemann–Liouville type and $u^{ext}(x, t) = (1 - x)^{6+9/17}t^{6+2/3}$.

6.2. Linear FPDEs with Riesz derivatives

We consider two linear FPDEs with Riesz space-fractional derivatives of field-variable orders, namely as space- and time-fractional diffusion and advection–diffusion problems.

Example 6.3. Fractional diffusion equation:

$${}^{RL}\mathcal{D}_t^{\zeta(x, t)} u = \frac{\partial^{1+\nu(t)}}{\partial |x|^{1+\nu(t)}} u + f(x, t) \quad (x, t) \in [-1, 1] \times [0, T], \quad (73)$$

$$u(\pm 1, t) = 0, \quad (74)$$

$$u(x, 0) = 0. \quad (75)$$

Here, we examine the same linear and hyperbolic tangent $\zeta(x, t)$ and $\nu(t)$ as in previous section. Hence, by setting $\tau = \mu = 1/2$ as the average values for the aforementioned field-variable order, we seek the solution to (73) to be of the form (33). Next, we substitute $u_N(x, t)$ in (4), and require again the corresponding residual to vanish at (x_i, t_k) to construct

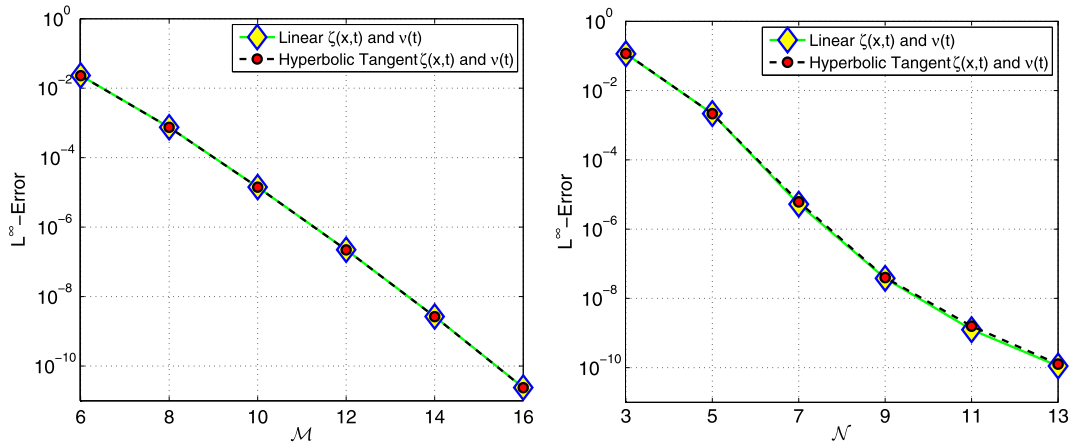


Fig. 4. Time- and space-fractional linear diffusion equation with Riesz spatial operators: (left) spatial p -refinement, and (right) temporal p -refinement. The exact solution is given by the fractional $u^{ext}(x, t) = \sin(\pi x)t^{6+2/3}$, where $(x, t) \in [-1, 1] \times [0, 2]$; moreover, the temporal and spatial fractional orders are respectively taken as the following field-variable functions, denoted as linear $\zeta(x, t) = (\frac{5+4x}{10})(\frac{1+4t}{10})$ and $1 + \nu(t) = 1 + \frac{1+4t}{10}$, also hyperbolic tangent $\zeta(x, t) = [1 + \tanh(x)][1 + \tanh(t - 1)]/4$ and $1 + \nu(t) = [3 + \tanh(t - 1)]/2$.

$${}^{Riesz}S_D \vec{u} = \vec{b}, \quad (76)$$

where ${}^{Riesz}S_D$ is an $(\mathcal{M} - 1)(\mathcal{N} - 1) \times (\mathcal{M} - 1)(\mathcal{N} - 1)$ matrix, whose entries are given by

$${}^{Riesz}S_D\{I_{ik}, J_{mn}\} = \{{}^{RL}\mathbf{D}_t^\zeta\}_{ikn}\delta_{im} - \{\mathbf{D}_{Riesz}^{1+\nu}\}_{ikm}\delta_{kn}, \quad (77)$$

where $\{\mathbf{D}_{Riesz}^{1+\nu}\}_{ikm}$ is given in (F.5), in which $i, m = 2, 3, \dots, \mathcal{M}$ and $k, n = 2, 3, \dots, \mathcal{N}$. We note that in the nodal expansion (33), we consider $\mathcal{M} + 1$, rather than $\mathcal{M}$ in the FLI (33), collocation points in the spatial dimension.

In a similar fashion, we plot the corresponding spatial and temporal p -refinement in Fig. 4, where we consider the exact solution this time to be $u^{ext}(x, t) = \sin(\pi x)t^{6+2/3}$, where we observe the exponential-like decay of the error versus the expansion order in each case.

Example 6.4. Fractional advection–diffusion equation:

$${}^{RL}D_t^{\zeta(x,t)}u + \frac{\partial^{\sigma(x,t)}}{\partial|x|^{\sigma(x,t)}}u = \frac{\partial^{1+\nu(t)}}{\partial|x|^{1+\nu(t)}}u + f(x, t) \quad (x, t) \in [-1, 1] \times [0, 2], \quad (78)$$

$$u(\pm 1, t) = 0, \quad (79)$$

$$u(x, 0) = 0. \quad (80)$$

We seek the solution to (78) to be of the form (33) and substitute it in (78). Then, we require the corresponding residual to vanish at (x_i, t_k) to obtain

$${}^{Riesz}S_{AD} \vec{u} = \vec{b}, \quad (81)$$

where the entries of ${}^{Riesz}S_{AD}$ are given by

$${}^{Riesz}S_{AD}\{I_{ik}, J_{mn}\} = \{{}^{RL}\mathbf{D}_t^\zeta\}_{ikn}\delta_{im} + (\{\mathbf{D}_{Riesz}^\sigma\}_{ikm} - \{\mathbf{D}_{Riesz}^{1+\nu}\}_{ikm})\delta_{kn}, \quad (82)$$

where $\{\mathbf{D}_{Riesz}^\sigma\}_{ikm}$ is given in (E.3), in which $i, m = 2, 3, \dots, \mathcal{M}$ and $k, n = 2, 3, \dots, \mathcal{N}$. Now, given the linear and hyperbolic tangent $\zeta(x, t)$ and $\nu(t)$ as in previous case, we set $\tau = \mu = 1/2$ as the average values for the aforementioned field-variable order.

In Fig. 5, we present the exponential-like decay of L^∞ -error in the corresponding spatial and temporal p -refinement. The exact solution is again given by the fractional in-time function $u^{ext}(x, t) = \sin(\pi x)t^{6+2/3}$. As before, the temporal and spatial fractional orders are respectively taken as the following field-variable functions, denoted as linear $\zeta(x, t) = \sigma(x, t) = (\frac{5+4x}{10})(\frac{1+4t}{10})$ and $1 + \nu(t) = 1 + \frac{1+4t}{10}$, also hyperbolic tangent $\zeta(x, t) = \sigma(x, t) = [1 + \tanh(x)][1 + \tanh(t - 1)]/4$ and $1 + \nu(t) = [3 + \tanh(t - 1)]/2$.

6.3. A penalty method for FPDEs

We consider the following variable-order in time and space diffusion equation subject to the initial condition $u(x, 0) = g(x)$ and in absence of the any external forcing term.

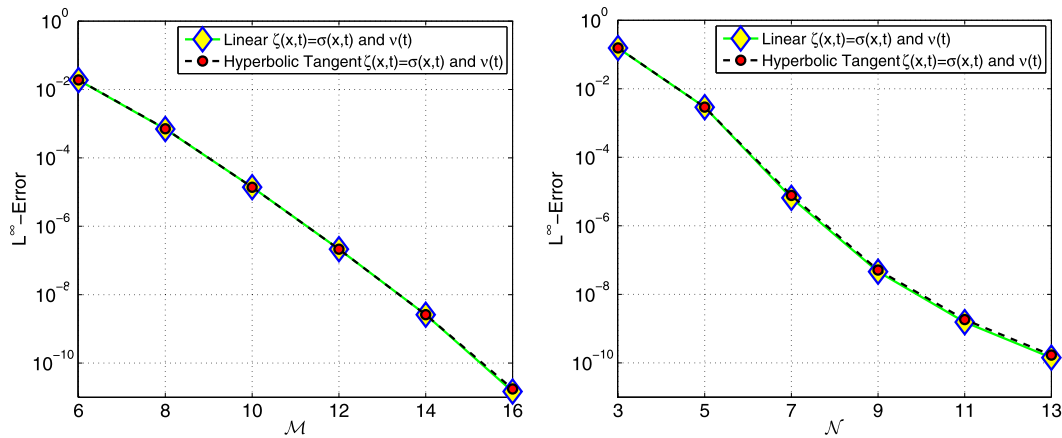


Fig. 5. Time- and space-fractional linear advection–diffusion equation with Riesz spatial operators: (left) spatial p -refinement, and (right) temporal p -refinement. The exact solution is given by the fractional $u^{\text{ext}}(x, t) = \sin(\pi x)t^{6+2/3}$, where $(x, t) \in [-1, 1] \times [0, 2]$; moreover, the temporal and spatial fractional orders are respectively taken as the following field-variable functions, denoted as linear $\zeta(x, t) = \sigma(x, t) = (\frac{5+4x}{10})(\frac{1+4t}{10})$ and $1 + \nu(t) = 1 + \frac{1+4t}{10}$, also hyperbolic tangent $\zeta(x, t) = \sigma(x, t) = [1 + \tanh(x)][1 + \tanh(t - 1)]/4$ and $1 + \nu(t) = [3 + \tanh(t - 1)]/2$.

Example 6.5.

$${}_0^C D_t^{\zeta(x,t)} u = \frac{\partial^{1+\nu(x,t)}}{\partial |x|^{1+\nu(x,t)}} u, \quad \forall (x, t) \in [-1, 1] \times [0, T], \quad (83)$$

$$u(\pm 1, t) = 0, \quad (84)$$

$$u(x, 0) = g(x), \quad (85)$$

In this example, $u(x, t)$ is assumed to be *continuous*, and we employ the time-derivative in (83) to be of Caputo type and the spatial derivative of Riesz type. To enforce the inhomogeneous initial condition $u(x, 0) = g(x)$, we present the following *penalty method* as follows: find $u_N(x, t) \approx u(x, t)$ such that

$${}_0^C D_t^{\zeta(x,t)} u_N = \frac{\partial^{1+\nu(x,t)}}{\partial |x|^{1+\nu(x,t)}} u_N - \mathcal{E} Q^-(t) [u_N(x, 0) - g(x)], \quad (86)$$

$$u(\pm 1, t) = 0, \quad (87)$$

where $Q^-(0) = 1$ and it vanishes at the rest of temporal collocation points $t \in (0, T]$. Next, we seek the solution u_N of the form

$$u_N(x, t) = \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}+1} u_N(x_m, t_n) h_m(x) h_n(t), \quad (88)$$

where $h_n(t)$ are the standard Legendre polynomials defined in $[0, T]$. This scheme is consistent since as $u_N \rightarrow u$ the penalty term vanishes asymptotically. Moreover, the global (spectral) treatment of the fractional time-derivative results in the penalty method in (86) to be *unconditionally stable* when $\mathcal{E} > 0$, confirmed by our extensive numerical experiments. In fact, the bigger the penalty coefficient $\mathcal{E}$, the stronger enforcement of the initial condition is achieved. Hence, we set $\mathcal{E} = 10^{15}$, which enforces the initial condition up to the machine precision. The dimension of the problem then becomes $(\mathcal{M} - 1) \times (\mathcal{N} + 1)$ since $u_N(\pm 1, t_n) = 0$.

By substituting (88) in (83) and require the residual to vanish at the collocation points we obtain the following linear system

$${}^{CR} \mathcal{S}_D \vec{u} = \vec{b}, \quad (89)$$

in which

$${}^{CR} \mathcal{S}_D \{I_{ik}, J_{mn}\} = \{ {}^C \mathbf{D}_t^{\zeta} \}_{ikn} \delta_{im} - \{ \mathbf{D}_{Riesz}^{1+\nu} \}_{ikm} \delta_{kn} + \mathcal{E} \delta_{mn} \delta_{1k}, \quad (90)$$

and

$$\vec{b}\{I_{ik}\} = -\mathcal{E} g(x_i) \delta_{1k}, \quad (91)$$

where $\{ {}^C \mathbf{D}_t^{\zeta} \}_{ikn}$ is given in (57), in which $i, m = 2, 3, \dots, \mathcal{M}$ and $k, n = 1, 2, \dots, \mathcal{N} + 1$. Here, the prescript *CR* recalls the Caputo and Riesz derivatives employed in the temporal and spatial dimensions, respectively.

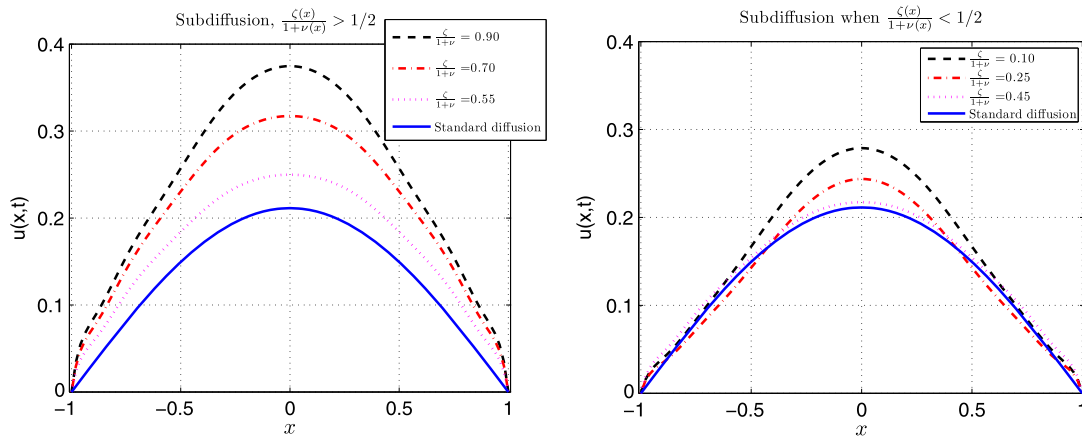


Fig. 6. Variable-order diffusion problem: the initial condition is $u(x, 0) = (1 - x^2)^4$ and the solutions are obtained at $t = \frac{1}{2}$, where the space-fractional order is $(1 + \nu(x)) \in (1, 2)$ and the time-fractional order is $\zeta(x) \in (0, 1)$, defined as spatial functions, where the ratio $\zeta(x)/(1 + \nu(x))$ is greater than $1/2$ (left), and is smaller than $1/2$ (right). In these test-cases, the fractional orders $\zeta(x)$ and $\nu(x)$ are given as constant when $x \in [-1/2, 1/2]$ and they vary linearly towards the boundaries, such that they keep the ratio invariant.

In Fig. 6, we solve (86) subject to the initial condition is $u(x, 0) = (1 - x^2)^4$ and we plot the results at $t = \frac{1}{2}$, where the space-fractional order is $(1 + \nu(x)) \in (1, 2)$ and the time-fractional order is $\zeta(x) \in (0, 1)$, defined as spatial functions, where the ratio $\zeta(x)/(1 + \nu(x))$ is greater than $1/2$ (left), and is smaller than $1/2$ (right). Here, we keep the fractional orders $\zeta(x)$ and $\nu(x)$ constant when $x \in [-1/2, 1/2]$ and allow them to decay linearly towards the boundaries, such that they keep the ratio invariant. The left plot reveals the sub-diffusion decay of the initial solution as expected. However, the right plot exhibits some *local super-diffusive* effects near the boundaries unexpectedly, which translates into faster decay compared to the standard diffusion. It is in contrast to the sub-diffusive nature of the problem when $\zeta \in (0, 1)$. Such a local effect can be associated with the way we define $\zeta(x)$ and $\nu(x)$, which is only x -dependent when $x \in [-1, -1/2]$ and $x \in [1/2, 1]$ and is constant in the middle domain. The polynomial order in time i.e., $\mathcal{N}$ and in space $\mathcal{M}$ are set to 18 in all simulations.

6.4. Nonlinear FPDEs

Here, we consider the nonlinear inviscid and viscous Burgers equation with field-variable orders in space.

Example 6.6. Inviscid and viscous Burgers' equation:

$$\frac{\partial u}{\partial t} + u^* \mathcal{D}_x^{\sigma(x,t)} u = K^* \mathcal{D}_x^{1+\nu(x,t)} u + f(u; x, t), \quad (92)$$

$$u(\pm 1, t) = 0, \quad (93)$$

$$u(x, 0) = 0. \quad (94)$$

Depending on the type of the fractional derivatives $^* \mathcal{D}_x$, either of left-sided Riemann–Liouville or Riesz type, we seek the solution to (92) to be of the form

$$u_N(x, t_k) = \sum_{m=2}^{\mathcal{M}-1} u_N(x_m, t_k) L_m^\mu(x), \quad (95)$$

when $^* \mathcal{D}_x \equiv {}^{RL} \mathcal{D}_x$, given the homogeneous boundary conditions. Alternatively, we seek the solution to (92) to be of the form

$$u_N(x, t_k) = \sum_{m=2}^{\mathcal{M}} u_N(x_m, t_k) h_m(x), \quad (96)$$

when $^* \mathcal{D}_x$ is of Riesz type. In this case, we employ a third-order Adams–Bashforth scheme to carry out the time-integration of (92), where we partition the time interval $[0, T]$ into equidistant points such that $\Delta t = T/N_g$, in which N_g denotes the number of time-grid-points. Denoting by $\vec{u}_N^j$ the vector of solution at time $t_j = j \cdot \Delta t$, we obtain the following fully discrete form

$$\frac{\vec{u}_N^{k+1} - \vec{u}_N^k}{\Delta t} = \sum_{q=1}^{\mathcal{J}} \alpha_q \left([-\text{diag}(\vec{u}_N^{k+1-q}) \cdot \mathbf{D}^\sigma + K \mathbf{D}^{1+\nu}] \cdot \vec{u}_N^{k+1-q} + \vec{f}^{k+1-q} \right), \quad (97)$$

Table 1

L^∞ -norm error of the numerical solution to (92) with $\mathcal{M}$, corresponding to the fractional orders $\sigma(x, t) = \nu(x, t) = (\frac{5+4x}{10})(\frac{1+4t}{10})$, hence, we set $\mu = 1/2$, the mean-value. The top table corresponds to the case where left-sided Riemann–Liouville fractional derivatives are employed. In this case, the exact solution is $u^{\text{ext}}(x, t) = t^3(1+x)^{6+2/3}$, $x \in [-1, 1]$ and we set $\Delta t = 1/200$. The bottom table corresponds to the case where Riesz fractional derivatives are used and the exact solution is $u^{\text{ext}}(x, t) = t^3 \sin(\pi x)$, $x \in [-1, 1]$ and we set $\Delta t = 1/600$. In both cases, the simulation time $T = 1$, where in the third-order Adams–Bashforth time-integration scheme.

$\mathcal{M}$	Inviscid Burgers ($K = 0$)	Viscous Burgers ($K = 1$)
<i>Left-sided Riemann–Liouville</i> ${}^*\mathcal{D}_x \equiv {}^{RL}\mathcal{D}_x$		
5	1.88×10^{-1}	2.73×10^{-1}
7	8.12×10^{-4}	8.82×10^{-3}
9	1.44×10^{-5}	1.38×10^{-5}
<i>Riesz fractional derivative</i> ${}^*\mathcal{D}_x \equiv \frac{\partial}{\partial x }$		
5	3.40×10^{-2}	1.42×10^{-1}
7	3.93×10^{-3}	9.68×10^{-3}
9	3.65×10^{-4}	6.32×10^{-4}

where $\tilde{f}^{k+1-q} = f(\tilde{x}, t_{k+1-q})$, $\mathcal{J}$ represents the order of the method and α_q are the coefficients of the $\mathcal{J}$ -th order Adams–Bashforth method.

Remark 6.7. The differentiation matrices $\mathbf{D}^\sigma$ and $\mathbf{D}^{1+\nu}$ are now two-dimensional matrices. However, we note that the entries of these matrices must be updated in each iteration due to the temporal variability in the fractional orders $\sigma(x, t)$ and $\nu(x, t)$. To this end, we use (37) and (41) when ${}^*\mathcal{D}_x \equiv {}^{RL}\mathcal{D}_x$, where we substitute t_k by $(k+1-q)\Delta t$. In addition, when ${}^*\mathcal{D}_x$ is of Riesz type, we construct the two-dimensional matrices $\mathbf{D}^\sigma$ and $\mathbf{D}^{1+\nu}$ using (52) and (53) similarly by substituting t_k by $(k+1-q)\Delta t$ in each iteration.

In Table 1, we demonstrate the exponential decay of L^∞ -norm error of the numerical solution to (92) with $\mathcal{M}$, where left-sided Riemann–Liouville and Riesz fractional derivatives are employed. Here we examine field-variable fractional orders $\sigma(x, t) = \nu(x, t) = (\frac{5+4x}{10})(\frac{1+4t}{10})$. When Riemann–Liouville derivatives are utilized, we set the exact solution to $u^{\text{ext}}(x, t) = t^3(1+x)^{6+2/3}$ and $\Delta t = 1/200$, and in the case of the Riesz fractional derivatives we set the exact solution $u^{\text{ext}}(x, t) = t^3 \sin(\pi x)$ while $\Delta t = 1/600$. For both cases, the simulation time $T = 1$ and we employ a third-order Adams–Bashforth time-integration scheme.

7. Summary and discussion

In this paper, we have developed a highly-accurate fractional spectral collocation method for solving linear and nonlinear FPDEs with field-variable temporal and/or spatial fractional orders. To this end and corresponding to the type of spatial derivatives (either Riemann–Liouville or Riesz), we introduced a new family of interpolants, called *left-/right-sided and central fractional Lagrange interpolants*, which satisfy the Kronecker delta property at collocation points. We constructed such interpolators to approximate the aforementioned fractional operators of both (left-/right-sided) Riemann–Liouville and Riesz type. We obtained the corresponding fractional differentiation matrices exactly. We solved several variable-order FPDEs including time- and space-fractional advection–equation, time- and space-fractional advection–diffusion equation, and finally the space-fractional Burgers’ equation. We also developed an unconditionally stable penalty method that efficiently treats FPDEs subject to inhomogeneous initial conditions. In our numerical examples, we demonstrated the exponential decay of L^∞ -error in the aforementioned model problems.

In addition to the accuracy, the key idea to the efficiency in our approach was to collocate the field-variable orders, as well as the resulting fractional FPDEs, at the collocation points. In fact, the C^0 -continuity of the fractional order in fixed-order FDPEs allowed us to require only the C^0 -continuity for the temporally and/or spatially-variable fractional orders in this study. This assumption made the point-wise evaluation of such field-variable orders at any arbitrary point well-defined in the space–time domain, hence, led us to circumvent the need for any quadrature rules that usually arise in spectral methods. To our experience, such numerical integrations become significantly costly when Galerkin/Petrov–Galerkin methods are employed, moreover, the resulting weak forms yield non-separable linear systems, which may lead to prohibitive computations in high dimensions.

Acknowledgements

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Appendix A. Proof of Theorem 5.1 (* $\mathcal{D}_x^{\sigma(x,t)} \equiv {}^{RL}_a \mathcal{D}_x^{\sigma(x,t)}$)

We substitute (29) in (28) and take the $\sigma(x, t)$ -th order fractional derivative. We do this by mapping the interval $x \in [a, b]$ to the standard domain $\xi \in [-1, 1]$ through $x(\xi) = \frac{b+a}{2} + \frac{b-a}{2}\xi$ and following (10) as

$$\begin{aligned} {}^{RL}_a \mathcal{D}_x^{\sigma(x,t)} u_N &= \left(\frac{2}{b-a} \right)^{\sigma(x(\xi),t)} {}_{-1} \mathcal{D}_\xi^{\sigma(x(\xi),t)} u_N(x(\xi), t) \\ &= \left(\frac{2}{b-a} \right)^{\sigma} {}_{-1} {}^{RL} \mathcal{D}_\xi^{\sigma} \left\{ \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) L_m^\mu(x(\xi)) \mathcal{T}_n^\tau(t) \right\} \\ &= \left(\frac{2}{b-a} \right)^{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) {}_{-1} {}^{RL} \mathcal{D}_\xi^{\sigma} \{ L_m^\mu(x(\xi)) \} \mathcal{T}_n^\tau(t) \\ &= \left(\frac{2}{b-a} \right)^{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) {}_{-1} {}^{RL} \mathcal{D}_\xi^{\sigma} \left\{ \left(\frac{\xi - \xi_1}{\xi_m - \xi_1} \right)^\mu \prod_{\substack{k=1 \\ k \neq m}}^{\mathcal{M}} \left(\frac{\xi - \xi_k}{\xi_m - \xi_k} \right) \right\} \mathcal{T}_n^\tau(t) \\ &= \left(\frac{2}{b-a} \right)^{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) {}_{-1} {}^{RL} \mathcal{D}_\xi^{\sigma} \{ (1 + \xi)^\mu \mathcal{G}_m(\xi) \} A_m \mathcal{T}_n^\tau(t) \end{aligned}$$

where $(2/(b-a))^{\sigma(x,t)}$ is a strictly positive function, $A_m = 1/(\xi_m + 1)^\mu$ and $\mathcal{G}_m(\xi) = \prod_{\substack{k=1 \\ k \neq m}}^{\mathcal{M}} \left(\frac{\xi - \xi_k}{\xi_m - \xi_k} \right)$, $m = 2, 3, \dots, \mathcal{M}$, are all polynomials of order $(\mathcal{M} - 1)$, which can be represented exactly in terms of Jacobi polynomials $P_{n-1}^{-\mu, \mu}(\xi)$ by

$$\mathcal{G}_m(\xi) = \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L P_{j-1}^{-\mu, \mu}(\xi). \quad (\text{A.1})$$

We note that the unknown coefficient matrix β_{mj}^L can be obtained analytically. The superscript L actually refers to the proper change of basis to the regular eigenfunctions of FSLP (25) whose left-sided fractional derivative is given exactly in (27). Next, by plugging (A.1) we obtain

$$\begin{aligned} {}^{RL}_a \mathcal{D}_x^{\sigma(x,t)} u_N &= \left(\frac{2}{b-a} \right)^{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) {}_{-1} {}^{RL} \mathcal{D}_\xi^{\sigma} \left\{ (1 + \xi)^\mu \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L P_{j-1}^{-\mu, \mu}(\xi) \right\} A_m \mathcal{T}_n^\tau(t) \\ &= \left(\frac{2}{b-a} \right)^{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L {}_{-1} {}^{RL} \mathcal{D}_\xi^{\sigma} \{ (1 + \xi)^\mu P_{j-1}^{-\mu, \mu}(\xi) \} A_m \mathcal{T}_n^\tau(t) \end{aligned}$$

Hence, by (27) we obtain

$${}^{RL}_a \mathcal{D}_x^{\sigma(x,t)} u_N = \left(\frac{2}{b-a} \right)^{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) A_m \mathcal{T}_n^\tau(t) \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L {}^{RL} \mathcal{D}_\xi^{\sigma(1)} \mathcal{P}_j^\mu(\xi). \quad (\text{A.2})$$

Case A-I (Constant $\sigma = \mu \in (0, 1)$). We use the property (27) and obtain

$${}^{RL}_a \mathcal{D}_x^{\sigma} u_N = \left(\frac{2}{b-a} \right)^{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) A_m \mathcal{T}_n^\tau(t) \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \frac{\Gamma(j + \sigma)}{\Gamma(j)} P_{j-1}(\xi). \quad (\text{A.3})$$

Consequently, by evaluating ${}_a \mathcal{D}_x^{\sigma} u_N(x, t)$ at the collocation points (x_i, t_k) and recalling that $L_n^\tau(t_k) = \delta_{kn}$, we obtain

$$\begin{aligned} {}^{RL}_a \mathcal{D}_x^{\sigma} u_N(x, t)|_{(x_i, t_k)} &= \left(\frac{2}{b-a} \right)^{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) A_m \delta_{kn} \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \frac{\Gamma(j + \sigma)}{\Gamma(j)} P_{j-1}(\xi_i) \\ &= \sum_{m=2}^{\mathcal{M}} \{ {}^{RL} \mathbf{D}_L^{\sigma} \}_{im} u_N(x_m, t_k), \end{aligned}$$

where $\{^{RL}\mathbf{D}_L^\sigma\}_{im}$ are the entries of the $(\mathcal{M}-1) \times (\mathcal{M}-1)$ left-sided spatial differentiation matrix $^{RL}\mathbf{D}_L^\sigma$ of Riemann–Liouville type, given by

$$\{^{RL}\mathbf{D}_L^\sigma\}_{im} = \left(\frac{2}{b-a}\right)^\sigma A_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \frac{\Gamma(j+\sigma)}{\Gamma(j)} P_{j-1}(\xi_i). \quad (\text{A.4})$$

Case A-II (The general $\sigma(x, t) \in (0, 1)$). Following [50] and [1], we first represent the polyfractional basis $^{(2)}\mathcal{P}_j^\mu(\xi)$ in terms of a sum and take a left-sided Riemann–Liouville fractional derivative, to obtain $F_j^{L,\sigma}$ as

$$\begin{aligned} F_j^{L,\sigma}(x(\xi), t) &\equiv {}_{-1}^{RL}\mathcal{D}_\xi^\sigma \{^{(1)}\mathcal{P}_j^\mu(\xi)\} \\ &= {}_{-1}^{RL}\mathcal{D}_\xi^\sigma \left\{ \sum_{q=0}^{j-1} \left(\frac{1}{2}\right)^q (-1)^{q+j-1} b_{jq}^\mu (1+\xi)^{q+\mu} \right\} \\ &= \sum_{q=0}^{j-1} \left(\frac{1}{2}\right)^q (-1)^{q+j-1} b_{jq-1}^* {}_{-1}^{RL}\mathcal{D}_\xi^\sigma \{(1+\xi)^{q+\mu}\}, \end{aligned}$$

where $b_{jq}^* = \binom{j-1+q}{q} \binom{j-1+\mu}{j-1-q}$. Now, we obtain $F_j^{L,\sigma}(x(\xi), t)$ exactly using (8) as

$$F_j^{L,\sigma}(x(\xi), t) = \sum_{q=0}^{j-1} b_{jq}^\mu (1+\xi)^{q+\mu-\sigma}, \quad (\text{A.5})$$

which leads to

$${}_{-a}^{RL}\mathcal{D}_x^{\sigma(x,t)} u_N(x, t) = \left(\frac{2}{b-a}\right)^{\sigma(x,t)} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) A_m \mathcal{T}_n^\tau(t) \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L F_j^{L,\sigma}(x, t), \quad (\text{A.6})$$

where

$$b_{jq}^\mu = \left(\frac{1}{2}\right)^q (-1)^{q+j-1} \binom{j-1+q}{q} \binom{j-1+\mu}{j-1-q} \frac{\Gamma(q+\mu+1)}{\Gamma(q+\mu+1-\sigma)}. \quad (\text{A.7})$$

Now, by evaluating ${}_{-a}^{RL}\mathcal{D}_x^{\sigma(x,t)} u_N(x, t)$ at the collocation points (x_i, t_k) also by $L_n^\tau(t_k) = \delta_{kn}$, we obtain

$$\begin{aligned} {}_{-a}^{RL}\mathcal{D}_x^{\sigma(x,t)} u_N|_{(x_i, t_k)} &= \left(\frac{2}{b-a}\right)^{\sigma(x_i, t_k)} \sum_{m=2}^{\mathcal{M}} u_N(x_m, t_k) A_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L F_j^{L,\sigma}(x_i, t_k) \\ &= \sum_{m=2}^{\mathcal{M}} \{^{RL}\mathbf{D}_L^\sigma\}_{imk} u_N(x_m, t_k), \end{aligned}$$

where $\{^{RL}\mathbf{D}_L^\sigma\}_{ikm}$ are the entries of the $(\mathcal{M}-1) \times (\mathcal{N}-1) \times (\mathcal{M}-1)$ left-sided spatial fractional differentiation matrix $^{RL}\mathbf{D}_L^\sigma$ of Riemann–Liouville sense, computed as

$$\{^{RL}\mathbf{D}_L^\sigma\}_{ikm} = \left(\frac{2}{b-a}\right)^{\sigma(x_i, t_k)} A_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L F_j^{L,\sigma}(x_i, t_k) \quad (\text{A.8})$$

where by re-arrangement $A_m = \left(\frac{b-a}{2x_m-2a}\right)^\mu$. It completes the proof.

Appendix B. Proof of Theorem 5.3 ($*\mathcal{D}_x^{1+\nu(x,t)} \equiv {}_{-a}^{RL}\mathcal{D}_x^{1+\nu(x,t)}$)

We use the sequential property (6) and switch the order of $\partial/\partial x$ and ${}_{-a}^{RL}\mathcal{D}_x^{\nu(x,t)}$ to avoid the difficulties arising from taking the first partial derivative of the corresponding Euler gamma functions with respect to x . Hence,

$$\begin{aligned}
{}^{RL}_a \mathcal{D}_x^{1+\nu(x,t)} u_N(x, t) &= {}^{RL}_a \mathcal{D}_x^{\nu(x,t)} \frac{\partial}{\partial x} \{u_N(x, t)\} \\
&= \left(\frac{2}{b-a}\right)^{\nu(x,t)} {}^{RL}_{-1} \mathcal{D}_\xi^{\nu(x,t)} \left\{ \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) A_m \mathcal{T}_n^\tau(t) \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \left(\frac{2}{b-a}\right) \frac{\partial}{\partial \xi} [({}^{(1)}\mathcal{P}_j^\mu(\xi)) \right\} \\
&= \left(\frac{2}{b-a}\right)^{(1+\nu(x,t))} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) A_m \mathcal{T}_n^\tau(t) \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L {}^{RL}_{-1} \mathcal{D}_\xi^{\nu(x,t)} \left\{ \frac{\partial}{\partial \xi} [({}^{(1)}\mathcal{P}_j^\mu(\xi)) \right\}.
\end{aligned}$$

Now, by substituting (25) we obtain

$$\begin{aligned}
{}^{RL}_a \mathcal{D}_x^{1+\nu(x,t)} u_N(x, t) &= \left(\frac{2}{b-a}\right)^{(1+\nu(x,t))} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) A_m \mathcal{T}_n^\tau(t) \\
&\quad \cdot \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \left[(\mu) {}^{RL}_{-1} \mathcal{D}_\xi^{\nu(x,t)} \{(1+\xi)^{\mu-1} P_{j-1}^{-\mu, \mu}(\xi)\} \right. \\
&\quad \left. + \left(\frac{j}{2}\right) {}^{RL}_{-1} \mathcal{D}_\xi^{\nu(x,t)} \{(1+\xi)^\mu P_{j-2}^{1-\mu, 1+\mu}(\xi)\} \right]
\end{aligned} \tag{B.1}$$

in which the Jacobi polynomials $P_{j-1}^{-\mu, \mu}(\xi)$ and $P_{j-2}^{1-\mu, 1+\mu}(\xi)$ can be represented in terms of the following sums

$$\begin{aligned}
P_{j-1}^{-\mu, \mu}(\xi) &= \sum_{q=0}^{j-1} \left(\frac{1}{2}\right)^q (-1)^{q+j-1} b_{jq}^* (1+\xi)^q, \quad j \geq 1 \\
P_{j-2}^{1-\mu, 1+\mu}(\xi) &= \sum_{q=0}^{j-2} \left(\frac{1}{2}\right)^q (-1)^{q+j-2} B_{jq}^* (1+\xi)^q, \quad j \geq 2
\end{aligned}$$

respectively, where $B_{jq}^* = \binom{j+q}{q} \binom{j-1+\mu}{j-2-q}$. By substituting the Jacobi polynomials back into (B.1) and simplifying, it yields

$$\begin{aligned}
{}^{RL}_a \mathcal{D}_x^{1+\nu(x,t)} u_N(x, t) &= \left(\frac{2}{b-a}\right)^{1+\nu} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) A_m \mathcal{T}_n^\tau(t) \\
&\quad \cdot \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \left[(\mu) \mathbb{I}_{\{j \geq 1\}} \sum_{q=0}^{j-1} \left(\frac{1}{2}\right)^q (-1)^{q+j-1} b_{jq}^* {}^{RL}_{-1} \mathcal{D}_\xi^{\nu(x,t)} \{(1+\xi)^{q+\mu-1}\} \right. \\
&\quad \left. + \left(\frac{j}{2}\right) \mathbb{I}_{\{j \geq 2\}} \sum_{q=0}^{j-2} \left(\frac{1}{2}\right)^q (-1)^{q+j-2} B_{jq}^* {}^{RL}_{-1} \mathcal{D}_\xi^{\nu(x,t)} \{(1+\xi)^{q+\mu}\} \right].
\end{aligned}$$

Now, by virtue of (8), we exactly obtain the variable-order fractional derivative of u_N as

$$\begin{aligned}
{}^{RL}_a \mathcal{D}_x^{1+\nu(x,t)} u_N(x, t) &= \left(\frac{2}{b-a}\right)^{1+\nu} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) A_m \mathcal{T}_n^\tau(t) \\
&\quad \cdot \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \left[(\mu) \mathbb{I}_{\{j \geq 1\}} \sum_{q=0}^{j-1} \left(\frac{1}{2}\right)^q (-1)^{q+j-1} b_{jq}^\mu \frac{\Gamma(q+\mu)}{\Gamma(q+\mu-\nu(x,t))} (1+\xi)^{q+\mu-1-\nu} \right. \\
&\quad \left. + \left(\frac{j}{2}\right) \mathbb{I}_{\{j \geq 2\}} \sum_{q=0}^{j-2} \left(\frac{1}{2}\right)^q (-1)^{q+j-2} B_{jq}^\mu \frac{\Gamma(1+q+\mu)}{\Gamma(1+q+\mu-\nu(x,t))} (1+\xi)^{q+\mu-\nu} \right].
\end{aligned}$$

Next, by evaluating the above expression at the collocation points (x_i, t_k) , we obtain

$${}_a \mathcal{D}_x^{1+\nu(x,t)} u_N(x, t) \big|_{(x_i, t_k)} = \sum_{m=2}^{\mathcal{M}-1} \{ {}^{RL} \mathbf{D}_L^{1+\nu} \}_{ikm} u_N(x_m, t_k)$$

in which $\{ {}^{RL} \mathbf{D}_L^{1+\nu} \}_{ikm}$, when $\nu = \nu(x, t)$ are computed as

$$\{ {}^{RL}\mathbf{D}_L^{1+\nu} \}_{ikm} = \left(\frac{2}{b-a} \right)^{1+\nu(x_i, t_k)} A_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \mathcal{F}_j^{L, \nu}(x_i, t_k), \quad (\text{B.2})$$

where $i, m = 2, 3, \dots, \mathcal{M}$, $k = 2, 3, \dots, \mathcal{N}$, and $\mathcal{F}_j^{L, \nu}(x(\xi), t)$ is explicitly given by

$$\begin{aligned} \mathcal{F}_j^{L, \nu}(x(\xi), t) = & \mathbb{I}_{\{j \geq 1\}} \sum_{q=0}^{j-1} \mathbb{B}_{jq}^{\mu} \cdot (1 + \xi)^{q+\mu-1-\nu(x_i, t_k)} \\ & + \mathbb{I}_{\{j \geq 2\}} \sum_{q=0}^{j-2} B_{jq}^{\mu} \cdot (1 + \xi)^{q+\mu-\nu(x_i, t_k)}, \end{aligned} \quad (\text{B.3})$$

in which $j = 1, 2, \dots, \mathcal{M}$, and $A_m = \left(\frac{b-a}{2x_m-2a} \right)^{\mu}$. Finally $\mathbb{B}_{jq}^{\mu}$ and B_{jq}^{μ} are the corresponding expansion coefficients, obtained as

$$\mathbb{B}_{jq}^{\mu} = \mu \left(\frac{1}{2} \right)^q (-1)^{q+j-1} \binom{j-1+q}{q} \binom{j-1+\mu}{j-1-q} \frac{\Gamma(q+\mu)}{\Gamma(q+\mu-\nu(x_i, t_k))}, \quad (\text{B.4})$$

and

$$B_{jq}^{\mu} = \frac{j}{2} \left(\frac{1}{2} \right)^q (-1)^{q+j-2} \binom{j+q}{q} \binom{j-1+\mu}{j-2-q} \frac{\Gamma(1+q+\mu)}{\Gamma(1+q+\mu-\nu(x_i, t_k))}. \quad (\text{B.5})$$

Appendix C. Proof of Theorem 5.5 (* $\mathcal{D}_x^{\sigma(x, t)} \equiv {}^{RL}\mathcal{D}_b^{\sigma(x, t)}$)

We substitute (32) in (31) and take the $\sigma(x, t)$ -th order fractional derivative as

$$\begin{aligned} {}^{RL}\mathcal{D}_b^{\sigma(x, t)} u_N &= \left(\frac{2}{b-a} \right)^{\sigma(x(\xi), t)} \xi \mathcal{D}_1^{\sigma(x(\xi), t)} \{ u_N(x(\xi), t) \} \\ &= \left(\frac{2}{b-a} \right)^{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) {}^{RL}\mathcal{D}_1^{\sigma} \{ R_m^{\mu}(x(\xi)) \} \mathcal{T}_n^{\tau}(t) \\ &= \left(\frac{2}{b-a} \right)^{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) {}^{RL}\mathcal{D}_1^{\sigma} \left\{ \left(\frac{\xi \mathcal{M} - \xi}{\xi \mathcal{M} - \xi_m} \right)^{\mu} \prod_{\substack{k=1 \\ k \neq m}}^{\mathcal{M}} \left(\frac{\xi - \xi_k}{\xi_m - \xi_k} \right) \right\} \mathcal{T}_n^{\tau}(t) \\ &= \left(\frac{2}{b-a} \right)^{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) {}^{RL}\mathcal{D}_{-1}^{\sigma} \{ (1 - \xi)^{\mu} \mathcal{G}_m(\xi) \} \mathcal{A}_m \mathcal{T}_n^{\tau}(t) \end{aligned}$$

in which $\mathcal{A}_m = 1/(\xi \mathcal{M} - \xi_m)^{\mu}$ and this time we can re-represent $\mathcal{G}_m(\xi)$, $m = 2, 3, \dots, \mathcal{M}$, in terms of another set of Jacobi polynomials $P_{n-1}^{\mu, -\mu}(\xi)$ exactly by

$$\mathcal{G}_m(\xi) = \sum_{j=1}^{\mathcal{M}} \beta_{mj}^R P_{j-1}^{\mu, -\mu}(\xi), \quad (\text{C.1})$$

where the superscript R in (C.1) now refers to the change of basis to the regular eigenfunctions of FSLP of second kind (26) whose right-sided fractional derivative is given exactly in (27). We again highlight that the unknown coefficient matrix β_{mj}^R can be obtained analytically. Next, by plugging (A.1) we obtain

$$\begin{aligned} {}^{RL}\mathcal{D}_b^{\sigma(x, t)} u_N &= \left(\frac{2}{b-a} \right)^{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) {}^{RL}\mathcal{D}_1^{\sigma} \left\{ (1 - \xi)^{\mu} \sum_{j=1}^{\mathcal{M}} \beta_{mj}^R P_{j-1}^{\mu, -\mu}(\xi) \right\} \mathcal{A}_m \mathcal{T}_n^{\tau}(t) \\ &= \left(\frac{2}{b-a} \right)^{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \sum_{j=1}^{\mathcal{M}} \beta_{mj}^R {}^{RL}\mathcal{D}_1^{\sigma} \{ (1 - \xi)^{\mu} P_{j-1}^{\mu, -\mu}(\xi) \} \mathcal{A}_m \mathcal{T}_n^{\tau}(t). \end{aligned}$$

Hence, by (27)

$${}^{RL}\mathcal{D}_b^{\sigma(x, t)} u_N = \left(\frac{2}{b-a} \right)^{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \mathcal{A}_m \mathcal{T}_n^{\tau}(t) \sum_{j=1}^{\mathcal{M}} \beta_{mj}^R {}^{RL}\mathcal{D}_1^{\sigma} \{ {}^{(2)}\mathcal{P}_j^{\mu}(\xi) \}. \quad (\text{C.2})$$

Case C-I (Constant $\sigma = \mu \in (0, 1)$). By (27), we can directly obtain

$${}^{RL}\mathcal{D}_b^\sigma u_N = \left(\frac{2}{b-a}\right)^\sigma \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \mathcal{A}_m \mathcal{T}_n^\tau(t) \sum_{j=1}^{\mathcal{M}} \beta_{mj}^R \frac{\Gamma(j+\sigma)}{\Gamma(j)} P_{j-1}(\xi), \quad (C.3)$$

which we evaluate (x_i, t_k) to obtain,

$$\begin{aligned} {}^{RL}\mathcal{D}_b^\sigma u_N(x, t)|_{(x_i, t_k)} &= \left(\frac{2}{b-a}\right)^\sigma \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \mathcal{A}_m \delta_{kn} \sum_{j=1}^{\mathcal{M}} \beta_{mj}^R \frac{\Gamma(j+\sigma)}{\Gamma(j)} P_{j-1}(\xi_i) \\ &= \sum_{m=2}^{\mathcal{M}} \{ {}^{RL}\mathbf{D}_R^\sigma \}_{im} u_N(x_m, t_k), \end{aligned}$$

where $\{ {}^{RL}\mathbf{D}_R^\sigma \}_{im}$ are the entries of the $(\mathcal{M}-1) \times (\mathcal{M}-1)$ right-sided spatial differentiation matrix ${}^{RL}\mathbf{D}_R^\sigma$ of Riemann–Liouville sense, given by

$$\{ {}^{RL}\mathbf{D}_R^\sigma \}_{im} = \left(\frac{2}{b-a}\right)^\sigma \mathcal{A}_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^R \frac{\Gamma(j+\sigma)}{\Gamma(j)} P_{j-1}(\xi_i). \quad (C.4)$$

Case C-II (The general $\sigma(x, t) \in (0, 1)$). Following [1], we first represent the polyfractonomial basis ${}^{(2)}\mathcal{P}_j^\mu(\xi)$ in terms of the following sum then we take its right-sided Riemann–Liouville fractional derivative, denoted by $F_j^{R, \sigma}$, as

$$\begin{aligned} F_j^{R, \sigma}(x(\xi), t) &\equiv {}^{RL}\mathcal{D}_1^\sigma \{ {}^{(2)}\mathcal{P}_j^\mu(\xi) \} \\ &= {}^{RL}\mathcal{D}_1^\sigma \left\{ \sum_{q=0}^{j-1} \left(\frac{-1}{2}\right)^q b_{jq}^\mu (1-\xi)^{q+\mu} \right\} \\ &= \sum_{q=0}^{j-1} \left(\frac{-1}{2}\right)^q c_{jq}^* {}^{RL}\mathcal{D}_1^\sigma \{ (1-\xi)^{q+\mu} \}, \end{aligned}$$

where $c_{jq}^* = \binom{j-1+q}{q} \binom{j-1-\mu}{j-1-q}$. Now, we obtain $F_j^{R, \sigma}(x(\xi), t)$ exactly using (9) as

$$F_j^{R, \sigma}(x(\xi), t) = \sum_{q=0}^{j-1} c_{jq}^\mu (1-\xi)^{q+\mu-\sigma}, \quad (C.5)$$

where

$$c_{jq}^\mu = -\left(\frac{-1}{2}\right)^q \binom{j-1+q}{q} \binom{j-1-\mu}{j-1-q} \frac{\Gamma(q+\mu+1)}{\Gamma(q+\mu+1-\sigma)}, \quad (C.6)$$

which leads to

$${}^{RL}\mathcal{D}_b^{\sigma(x, t)} u_N(x, t) = \left(\frac{2}{b-a}\right)^{\sigma(x, t)} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \mathcal{A}_m \mathcal{T}_n^\tau(t) \sum_{j=1}^{\mathcal{M}} \beta_{mj}^R F_j^{R, \sigma}(x, t). \quad (C.7)$$

Now, by evaluating ${}^{RL}\mathcal{D}_b^{\sigma(x, t)} u_N(x, t)$ at the collocation points (x_i, t_k) we obtain

$$\begin{aligned} {}^{RL}\mathcal{D}_b^{\sigma(x, t)} u_N|_{(x_i, t_k)} &= \left(\frac{2}{b-a}\right)^{\sigma(x_i, t_k)} \sum_{m=2}^{\mathcal{M}} u_N(x_m, t_k) \mathcal{A}_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^R F_j^{R, \sigma}(x_i, t_k) \\ &= \sum_{m=2}^{\mathcal{M}} \{ {}^{RL}\mathbf{D}_R^\sigma \}_{imk} u_N(x_m, t_k), \end{aligned}$$

where $\{ {}^{RL}\mathbf{D}_R^\sigma \}_{imk}$ are the entries of the $(\mathcal{M}-1) \times (\mathcal{N}-1) \times (\mathcal{M}-1)$ right-sided spatial fractional differentiation matrix ${}^{RL}\mathbf{D}_R^\sigma$ of Riemann–Liouville sense, computed as

$$\{ {}^{RL}\mathbf{D}_R^\sigma \}_{imk} = \left(\frac{2}{b-a}\right)^{\sigma(x_i, t_k)} \mathcal{A}_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^R F_j^{R, \sigma}(x_i, t_k), \quad (C.8)$$

where by re-arrangement $\mathcal{A}_m = (\frac{b-a}{2b-2x_m})^\mu$. We recall from Remark 5.2 that if $\sigma = \sigma(x)$, we again reduce the dimension of ${}^{RL}\mathbf{D}_R^\sigma$ by one, hence, we can obtain the entries of the corresponding two-dimensional right-sided differentiation matrix as

$$\{{}^{RL}\mathbf{D}_R^\sigma\}_{im} = \left(\frac{2}{b-a}\right)^{\sigma(x_i)} \mathcal{A}_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L F_j^{R,\sigma}(x_i). \quad (\text{C.9})$$

Appendix D. Proof of Theorem 5.6 (* $\mathcal{D}_x^{\sigma(x,t)} \equiv {}^{RL}\mathcal{D}_b^{1+\nu(x,t)}$)

When the fractional order is both x - and t -dependent, to avoid the difficulties in computations of the first partial derivative in the corresponding gamma functions with respect to x , we alternatively write ${}^{RL}\mathcal{D}_b^{1+\nu(x,t)}$ as

$$\begin{aligned} {}^{RL}\mathcal{D}_b^{1+\nu(x,t)} u_N(x, t) &= {}^{RL}\mathcal{D}_b^{\nu(x,t)} \frac{\partial}{\partial x} \{u_N(x, t)\} \\ &= \left(\frac{2}{b-a}\right)^\nu {}^{RL}\mathcal{D}_1^{\nu(x,t)} \left\{ \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \mathcal{A}_m \mathcal{T}_n^\tau(t) \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \left(\frac{2}{b-a}\right) \frac{\partial}{\partial \xi} [{}^{(2)}\mathcal{P}_j^\mu(\xi)] \right\} \\ &= \left(\frac{2}{b-a}\right)^{1+\nu} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \mathcal{A}_m \mathcal{T}_n^\tau(t) \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L {}^{RL}\mathcal{D}_1^{\nu(x,t)} \left\{ \frac{\partial}{\partial \xi} [{}^{(2)}\mathcal{P}_j^\mu(\xi)] \right\}. \end{aligned}$$

Now, by substituting (26) we obtain

$$\begin{aligned} {}^{RL}\mathcal{D}_b^{1+\nu(x,t)} u_N(x, t) &= \left(\frac{2}{b-a}\right)^{1+\nu} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \mathcal{A}_m \mathcal{T}_n^\tau(t) \\ &\quad \cdot \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \left[(-\mu) {}^{RL}\mathcal{D}_1^{\nu(x,t)} \{(1-\xi)^{\mu-1} P_{j-1}^{\mu,-\mu}(\xi)\} \right. \\ &\quad \left. + \left(\frac{j}{2}\right) {}^{RL}\mathcal{D}_1^{\nu(x,t)} \{(1-\xi)^\mu P_{j-2}^{1+\mu,1-\mu}(\xi)\} \right] \end{aligned} \quad (\text{D.1})$$

in which the Jacobi polynomials $P_{j-1}^{\mu,-\mu}(\xi)$ and $P_{j-2}^{1+\mu,1-\mu}(\xi)$ can be represented in terms of the following sums

$$\begin{aligned} P_{j-1}^{\mu,-\mu}(\xi) &= \sum_{q=0}^{j-1} \left(\frac{-1}{2}\right)^q c_{jq}^\mu \cdot (1-\xi)^q, \quad j \geq 1, \\ P_{j-2}^{1+\mu,1-\mu}(\xi) &= \sum_{q=0}^{j-2} \left(\frac{-1}{2}\right)^q B_{jq}^\mu \cdot (1-\xi)^q, \quad j \geq 2, \end{aligned}$$

respectively. Now, by substituting the Jacobi polynomials back into (D.1) and simplifying, it yields

$$\begin{aligned} {}^{RL}\mathcal{D}_b^{1+\nu(x,t)} u_N(x, t) &= \left(\frac{2}{b-a}\right)^{1+\nu} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \mathcal{A}_m \mathcal{T}_n^\tau(t) \\ &\quad \cdot \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \left[(-\mu) \mathbb{I}_{\{j \geq 1\}} \sum_{q=0}^{j-1} \left(\frac{-1}{2}\right)^q c_{jq}^* {}^{RL}\mathcal{D}_1^{\nu(x,t)} \{(1-\xi)^{q+\mu-1}\} \right. \\ &\quad \left. + \left(\frac{j}{2}\right) \mathbb{I}_{\{j \geq 2\}} \sum_{q=0}^{j-2} \left(\frac{-1}{2}\right)^q c_{jq}^* {}^{RL}\mathcal{D}_1^{\nu(x,t)} \{(1-\xi)^{q+\mu}\} \right], \end{aligned}$$

where $c_{jq}^* = \binom{j+q}{q} \binom{j-1+\mu}{j-2-q}$. Now, by virtue of (9), we exactly obtain the variable-order fractional derivative of u_N as

$$\begin{aligned}
{}^{RL}\mathcal{D}_b^{1+\nu(x,t)} u_N(x, t) &= \left(\frac{2}{b-a} \right)^{1+\nu} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \mathcal{A}_m \mathcal{T}_n^\tau(t) \\
&\cdot \sum_{j=1}^{\mathcal{M}} \beta_{mj}^L \left[\mathbb{I}_{\{j \geq 1\}} \sum_{q=0}^{j-1} \mathbb{C}_{jq}^\mu (1-\xi)^{q+\mu-1-\nu} \right. \\
&\quad \left. + \mathbb{I}_{\{j \geq 2\}} \sum_{q=0}^{j-2} \mathbb{C}_{jq}^\mu (1-\xi)^{q+\mu-\nu} \right],
\end{aligned}$$

where

$$\mathbb{C}_{jq}^\mu = (\mu) \left(\frac{-1}{2} \right)^q \binom{j-1+q}{q} \binom{j-1-\mu}{j-1-q} \frac{\Gamma(q+\mu)}{\Gamma(q+\mu-\nu(x,t))} \quad (\text{D.2})$$

and

$$\mathbb{C}_{jq}^\mu = \left(\frac{-j}{2} \right) \left(\frac{-1}{2} \right)^q \binom{j+q}{q} \binom{j-1+\mu}{j-2-q} \frac{\Gamma(1+q+\mu)}{\Gamma(1+q+\mu-\nu(x,t))}. \quad (\text{D.3})$$

Next, by evaluating the above expression at the collocation points (x_i, t_k) , we obtain

$$\{\mathbf{D}_R^{1+\nu}\}_{ikm} = \left(\frac{2}{b-a} \right)^{1+\nu(x_i, t_k)} \mathcal{A}_m \sum_{j=1}^{\mathcal{M}} \beta_{mj}^R \mathcal{F}_j^{R,\nu}(x_i, t_k), \quad (\text{D.4})$$

where $\mathcal{F}_j^{R,\nu}(x(\xi), t)$ is obtained as

$$\begin{aligned}
\mathcal{F}_j^{R,\nu}(x(\xi), t) &= \mathbb{I}_{\{j \geq 1\}} \sum_{q=0}^{j-1} \mathbb{C}_{jq}^\mu \cdot (1-\xi)^{q+\mu-1-\nu(x_i, t_k)} \\
&\quad + \mathbb{I}_{\{j \geq 2\}} \sum_{q=0}^{j-2} \mathbb{C}_{jq}^\mu \cdot (1-\xi)^{q+\mu-\nu(x_i, t_k)},
\end{aligned} \quad (\text{D.5})$$

which completes the proof.

Appendix E. Proof of Theorem 5.8 ($\ast \mathcal{D}_x^{\sigma(x,t)} \equiv \partial^{\sigma(x,t)} u / \partial |x|^{\sigma(x,t)}$)

We substitute (34) in (33) and take the $\sigma(x, t)$ -th order fractional derivative. Once again, we perform the affine mapping from $x \in [a, b]$ to $\xi \in [-1, 1]$ as before we obtain

$$\begin{aligned}
\frac{\partial^{\sigma(x,t)} u_N(x, t)}{\partial |x|^{\sigma(x,t)}} &= \left(\frac{2}{b-a} \right)^{\sigma(x(\xi), t)} C_{\sigma(x(\xi), t)} (-1 \mathcal{D}_\xi^\sigma u_N + \xi \mathcal{D}_1^\sigma u_N) \\
&= \left(\frac{2}{b-a} \right)^\sigma C_\sigma \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) (-1 \mathcal{D}_\xi^\sigma \{h_m(x(\xi))\} + \xi \mathcal{D}_1^\sigma \{h_m(x(\xi))\}) \mathcal{T}_n^\tau(t),
\end{aligned}$$

where $h_m(x(\xi)) \equiv h_m(\xi)$ are all polynomials of order $(\mathcal{M})$, which can be represented exactly in terms of Legendre polynomials $P_n(\xi)$ as

$$h_m(\xi) = \sum_{j=0}^{\mathcal{M}} \tilde{\beta}_{mj} P_j(\xi). \quad (\text{E.1})$$

We again note that the coefficient matrix $\tilde{\beta}_{mj}$ can be obtained analytically. By plugging (E.1) we obtain

$$\frac{\partial^{\sigma(x,t)} u_N(x, t)}{\partial |x|^{\sigma(x,t)}} = \left(\frac{2}{b-a} \right)^\sigma C_\sigma \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \sum_{j=0}^{\mathcal{M}} \tilde{\beta}_{mj} (-1 \mathcal{D}_\xi^\sigma \{P_j(\xi)\} + \xi \mathcal{D}_1^\sigma \{P_j(\xi)\}) \mathcal{T}_n^\tau(t),$$

where by exact evaluation of the left- and right-sided fractional derivatives of the Legendre polynomials in (50) and (51),

$$\frac{\partial^{\sigma(x,t)} u_N(x, t)}{\partial |x|^{\sigma(x,t)}} = \left(\frac{2}{b-a}\right)^{\sigma} C_{\sigma} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \\ \times \left\{ \sum_{j=\lceil \sigma(x,t) \rceil}^{\mathcal{M}} \tilde{\beta}_{mj} \frac{\Gamma(j+1)}{\Gamma(j-\sigma+1)} \{ (1+\xi)^{-\sigma} P_j^{\sigma, -\sigma}(\xi) + (1-\xi)^{-\sigma} P_j^{-\sigma, \sigma}(\xi) \} \right\} \mathcal{T}_n^{\tau}(t), \quad (\text{E.2})$$

which we evaluate at the collocation points $u_N(x_i, t_k)$ to obtain

$$\left. \frac{\partial^{\sigma(x,t)} u_N(x, t)}{\partial |x|^{\sigma(x,t)}} \right|_{(x_i, t_k)} = \sum_{m=2}^{\mathcal{M}} \{ \mathbf{D}_{\text{Riesz}}^{\sigma} \}_{ikm} u_N(x_m, t_k)$$

in which $\{ \mathbf{D}_{\text{Riesz}}^{\sigma} \}_{ikm}$ are the entries of the three dimensional Riesz spatial differentiation matrix of order $\sigma(x_i, t_k)$, $\mathbf{D}_{\text{Riesz}}^{\sigma}$, given by

$$\{ \mathbf{D}_{\text{Riesz}}^{\sigma} \}_{ikm} = \left[\left(\frac{2}{b-a}\right)^{\sigma(x,t)} C_{\sigma(x,t)} \right]_{(x_i, t_k)} \sum_{j=1}^{\mathcal{M}} \tilde{\beta}_{mj} \mathcal{Z}_j^{\sigma}(x_k, t_k), \quad (\text{E.3})$$

where

$$\mathcal{Z}_j^{\sigma}(x, t) = \frac{\Gamma(j+1)}{\Gamma(j-\sigma(x_i, t_k)+1)} [(1+\xi)^{-\sigma(x,t)} P_j^{\sigma, -\sigma}(\xi) + (1-\xi)^{-\sigma(x,t)} P_j^{-\sigma, \sigma}(\xi)],$$

in which we recall that $\xi = 2\frac{x-a}{b-a}$.

Appendix F. Proof of Theorem 5.9 ($\ast \mathcal{D}_x^{1+v(x,t)} \equiv \partial^{1+v(x,t)} u / \partial |x|^{1+v(x,t)}$)

We first take the fractional and then the first derivative of the solution to obtain

$$\frac{\partial^{1+v(x,t)} u_N(x, t)}{\partial |x|^{1+v(x,t)}} = C_{1+v} \{ {}^{RL}\mathcal{D}_x^{v(x,t)} + {}^{RL}\mathcal{D}_b^{v(x,t)} \} \left(\frac{\partial u_N}{\partial x} \right) \\ = \left(\frac{2}{b-a}\right)^{1+v} C_{1+v} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \mathcal{T}_n^{\tau}(t) \{ {}^{RL}\mathcal{D}_{\xi}^v + {}^{RL}\mathcal{D}_1^v \} \sum_{j=0}^{\mathcal{M}} \tilde{\beta}_{mj} \frac{\partial}{\partial \xi} \{ P_j(\xi) \} \\ = \left(\frac{2}{b-a}\right)^{1+v} C_{1+v} \sum_{m=2}^{\mathcal{M}} \sum_{n=1}^{\mathcal{N}} u_N(x_m, t_n) \mathcal{T}_n^{\tau}(t) \sum_{j=0}^{\mathcal{M}} \tilde{\beta}_{mj} \mathcal{W}_j^v, \quad (\text{F.1})$$

in which we evaluate $\mathcal{W}_j^v \equiv (\frac{j+1}{2}) \{ {}^{RL}\mathcal{D}_{\xi}^v + {}^{RL}\mathcal{D}_1^v \} P_{j-1}^{1,1}(\xi)$ exactly by representing the Jacobi polynomial $P_{j-1}^{1,1}(\xi)$ as the following two alternative forms

$$P_{j-1}^{1,1}(\xi) = \sum_{q=0}^{j-1} C_j (-1)^{q+j-1} \left(\frac{1}{2}\right)^q (1+\xi)^q \quad (\text{F.2})$$

$$= \sum_{q=0}^{j-1} C_j \left(\frac{-1}{2}\right)^q (1-\xi)^q \quad (\text{F.3})$$

where $C_j = \binom{j+1+q}{q} \binom{j}{j-1-q}$. Hence,

$$\mathcal{W}_j^v = \left(\frac{j+1}{2}\right) ({}^{RL}\mathcal{D}_x^{v(x,t)} + {}^{RL}\mathcal{D}_b^{v(x,t)}) \{ P_{j-1}^{1,1}(\xi) \} \\ = \left(\frac{j+1}{2}\right) {}^{RL}\mathcal{D}_x^{v(x,t)} \left\{ \sum_{q=0}^{j-1} C_j (-1)^{q+j-1} \left(\frac{1}{2}\right)^q (1+\xi)^q \right\} \\ + \left(\frac{j+1}{2}\right) {}^{RL}\mathcal{D}_b^{v(x,t)} \left\{ \sum_{q=0}^{j-1} C_j \left(\frac{-1}{2}\right)^q (1-\xi)^q \right\},$$

by substituting $P_{j-1}^{1,1}(\xi)$ by (F.2) when taking the left-sided and by (F.3) when taking the right-sided Riemann–Liouville fractional derivative, respectively. Now, by (8) and (9), we obtain

$$\mathcal{W}_j^\nu = \left(\frac{j+1}{2}\right) \sum_{q=\lceil \nu \rceil}^{j-1} c_j \frac{(-\frac{1}{2})^q \Gamma(q+1)}{\Gamma(q+1-\nu(x,t))} [(-1)^{j-1} (1+\xi)^{q-\nu(x,t)} - (1-\xi)^{q-\nu(x,t)}]. \quad (\text{F.4})$$

Next, by substituting (F.4) into (F.1), evaluating it at the collocation points (x_i, t_k) , and using the Kronecker delta property $\mathcal{T}_n^\tau(t_k) = \delta_{kn}$, we obtain

$$\begin{aligned} \frac{\partial^{1+\nu(x,t)} u_N(x, t)}{\partial |x|^{1+\nu(x,t)}} \Big|_{(x_i, t_k)} &= \left[\left(\frac{2}{b-a} \right)^{1+\nu} C_{1+\nu} \right]_{(x_i, t_k)} \sum_{m=2}^{\mathcal{M}} u_N(x_m, t_k) \sum_{j=1}^{\mathcal{M}} \tilde{\beta}_{mj} \mathcal{W}_j^\nu \\ &= \sum_{m=2}^{\mathcal{M}} \{ \mathbf{D}_{\text{Riesz}}^{1+\nu(x,t)} \}_{ikm} u_N(x_m, t_k), \end{aligned}$$

in which $\{ \mathbf{D}_{\text{Riesz}}^{1+\nu(x,t)} \}_{ikm}$ denotes the x - and t -dependent fractional diffusion differentiation matrix of Riesz type, give by

$$\{ \mathbf{D}_{\text{Riesz}}^{1+\nu(x,t)} \}_{ikm} = \left[\left(\frac{2}{b-a} \right)^{1+\nu} C_{1+\nu} \right]_{(x_i, t_k)} \sum_{j=1}^{\mathcal{M}} \tilde{\beta}_{mj} \mathcal{W}_j^\nu(x, t). \quad (\text{F.5})$$

References

- [1] M. Zayernouri, G.E. Karniadakis, Fractional Sturm–Liouville eigen-problems: theory and numerical approximations, *J. Comput. Phys.* 47 (3) (2013) 2108–2131.
- [2] D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, Application of a fractional advection–dispersion equation, *Water Resour. Res.* 36 (6) (2000) 1403–1412.
- [3] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, Imperial College Press, 2010.
- [4] W. Chester, Resonant oscillations in closed tubes, *J. Fluid Mech.* 18 (1) (1964) 44–64.
- [5] J.J. Keller, Propagation of simple non-linear waves in gas filled tubes with friction, *Z. Angew. Math. Phys.* 32 (2) (1981) 170–181.
- [6] N. Sugimoto, T. Kakutani, ‘Generalized Burgers’ equation’ for nonlinear viscoelastic waves, *Wave Motion* 7 (5) (1985) 447–458.
- [7] R.L. Magin, *Fractional Calculus in Bioengineering*, Begell House Inc., Redding, CT, 2006.
- [8] J.P. Bouchaud, A. Georges, Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications, *Phys. Rep.* 195 (4) (1990) 127–293.
- [9] R. Metzler, J. Klafter, The random walk’s guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.* 339 (1) (2000) 1–77.
- [10] R. Klages, G. Radons, I.M. Sokolov, *Anomalous Transport: Foundations and Applications*, Wiley–VCH, 2008.
- [11] N. Sugimoto, Burgers equation with a fractional derivative; hereditary effects on nonlinear acoustic waves, *J. Fluid Mech.* 225 (631–653) (1991) 4.
- [12] E. Barkai, R. Metzler, J. Klafter, From continuous time random walks to the fractional Fokker–Planck equation, *Phys. Rev. E* 61 (1) (2000) 132.
- [13] B. Henry, S. Wearne, Fractional reaction–diffusion, *Phys. A, Stat. Mech. Appl.* 276 (3) (2000) 448–455.
- [14] T. Komatsu, On stable-like processes, in: *Probability Theory and Mathematical Statistics*, Tokyo, 1995, 1995, pp. 210–219.
- [15] K. Kikuchi, A. Negoro, On Markov process generated by pseudo-differential operator of variable order, *Osaka J. Math.* 34 (2) (1997) 319–335.
- [16] C. Coimbra, Mechanics with variable-order differential operators, *Ann. Phys.* 12 (11–12) (2003) 692–703.
- [17] G. Cooper, D. Cowan, Filtering using variable order vertical derivatives, *Comput. Geosci.* 30 (5) (2004) 455–459.
- [18] C. Tseng, Design of variable and adaptive fractional order FIR differentiators, *Signal Process.* 86 (10) (2006) 2554–2566.
- [19] L.E. Ramirez, C.F. Coimbra, A variable order constitutive relation for viscoelasticity, *Ann. Phys.* 16 (7–8) (2007) 543–552.
- [20] H. Pedro, M. Kobayashi, J. Pereira, C. Coimbra, Variable order modeling of diffusive–convective effects on the oscillatory flow past a sphere, *J. Vib. Control* 14 (9–10) (2008) 1659–1672.
- [21] H.G. Sun, W. Chen, Y.Q. Chen, Variable-order fractional differential operators in anomalous diffusion modeling, *Phys. A, Stat. Mech. Appl.* 388 (21) (2009) 4586–4592.
- [22] C.-M. Chen, F. Liu, V. Anh, I. Turner, Numerical schemes with high spatial accuracy for a variable-order anomalous subdiffusion equation, *SIAM J. Sci. Comput.* 32 (4) (2010) 1740–1760.
- [23] C.M. Chen, F. Liu, V. Anh, I. Turner, Numerical simulation for the variable-order Galilei invariant advection diffusion equation with a nonlinear source term, *Appl. Math. Comput.* 217 (12) (2011) 5729–5742.
- [24] H. Zhang, F. Liu, M.S. Phanikumar, M.M. Meerschaert, A novel numerical method for the time variable fractional order mobile–immobile advection–dispersion model, *Comput. Math. Appl.* 66 (5) (2013) 693–701.
- [25] W. Chen, J. Zhang, J. Zhang, A variable-order time-fractional derivative model for chloride ions sub-diffusion in concrete structures, *Fract. Calc. Appl. Anal.* 16 (1) (2013) 76–92.
- [26] P. Zhuang, F. Liu, V. Anh, I. Turner, Numerical methods for the variable-order fractional advection–diffusion equation with a nonlinear source term, *SIAM J. Numer. Anal.* 47 (3) (2009) 1760–1781.
- [27] S. Shen, F. Liu, J. Chen, I. Turner, V. Anh, Numerical techniques for the variable order time fractional diffusion equation, *Appl. Math. Comput.* 218 (22) (2012) 10861–10870.
- [28] C. Chen, F. Liu, K. Burrage, Y. Chen, Numerical methods of the variable-order Rayleigh–Stokes problem for a heated generalized second grade fluid with fractional derivative, *IMA J. Appl. Math.* (2012) 1–21.
- [29] S. Shen, F. Liu, V. Anh, I. Turner, J. Chen, A characteristic difference method for the variable-order fractional advection–diffusion equation, *J. Appl. Math. Comput.* (2013) 1–16.
- [30] X. Zhao, Z. Sun, G.E. Karniadakis, Second-order approximations for variable order fractional derivatives: algorithms and applications, *Journal of Computational Physics*, Accepted.
- [31] Y. Lin, C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation, *J. Comput. Phys.* 225 (2) (2007) 1533–1552.
- [32] X. Li, C. Xu, A space–time spectral method for the time fractional diffusion equation, *SIAM J. Numer. Anal.* 47 (3) (2009) 2108–2131.
- [33] X. Li, C. Xu, Existence and uniqueness of the weak solution of the space–time fractional diffusion equation and a spectral method approximation, *Commun. Comput. Phys.* 8 (5) (2010) 1016.
- [34] M.M. Khader, On the numerical solutions for the fractional diffusion equation, *Commun. Nonlinear Sci. Numer. Simul.* 16 (6) (2011) 2535–2542.
- [35] C. Piret, E. Hanert, A radial basis functions method for fractional diffusion equations, *J. Comput. Phys.* (2012) 71–81.

- [36] E. Doha, A. Bhrawy, S. Ezz-Eldien, A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order, *Comput. Math. Appl.* 62 (5) (2011) 2364–2373.
- [37] A.H. Bhrawy, M.M. Al-Shomrani, A shifted Legendre spectral method for fractional-order multi-point boundary value problems, *Adv. Differ. Equ.* 2012 (1) (2012) 1–19.
- [38] M. Maleki, I. Hashim, M.T. Kajani, S. Abbasbandy, An adaptive pseudospectral method for fractional order boundary value problems, *Abstr. Appl. Anal.* 2012 (2012).
- [39] D. Baleanu, A. Bhrawy, T. Taha, Two efficient generalized Laguerre spectral algorithms for fractional initial value problems, *Abstr. Appl. Anal.* 2013 (2013).
- [40] A. Bhrawy, M. Alghamdia, A new Legendre spectral Galerkin and pseudo-spectral approximations for fractional initial value problems, *Abstr. Appl. Anal.* 2013 (2013) 306746 (10 pp.).
- [41] W. Deng, J. Hesthaven, Local discontinuous Galerkin methods for fractional diffusion equations, *ESAIM: Math. Model. Numer. Anal.* (2013) 1845–1864.
- [42] Q. Xu, J. Hesthaven, Stable multi-domain spectral penalty methods for fractional partial differential equations, *J. Comput. Phys.* 257 (2014) 241–258.
- [43] M. Zayernouri, G.E. Karniadakis, Exponentially accurate spectral and spectral element methods for fractional ODEs, *J. Comput. Phys.* 257-Part A (2014) 460–480.
- [44] M. Zayernouri, G.E. Karniadakis, Discontinuous spectral element methods for time- and space-fractional advection equations, *SIAM J. Sci. Comput.* 36 (4) (2014) B684–B707.
- [45] M. Zayernouri, W. Cao, Z. Zhang, G.E. Karniadakis, Spectral and discontinuous spectral element methods for fractional delay equations, *SIAM J. Sci. Comput.* 36 (6) (2014) B904–B929.
- [46] M. Zayernouri, G.E. Karniadakis, Fractional spectral collocation method, *SIAM J. Sci. Comput.* 36 (1) (2014) A40–A62.
- [47] M. Zayernouri, M. Ainsworth, G.E. Karniadakis, A unified Petrov–Galerkin spectral method for fractional PDEs, *Comput. Methods Appl. Mech. Eng.* (2014), <http://dx.doi.org/10.1016/j.cma.2014.10.051>.
- [48] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, USA, 1999.
- [49] R. Askey, J. Fitch, Integral representations for Jacobi polynomials and some applications, *J. Math. Anal. Appl.* 26 (1969) 411–437.
- [50] G. Szegő, *Orthogonal Polynomials*, vol. 23, AMS Bookstore, 1992.