Journal of Computational and Applied Mathematics 245 (2013) 182–193

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On exponential mean-square stability of two-step Maruyama methods for stochastic delay differential equations

Wanrong Cao [a,b,*], Zhongqiang Zhang [b]

[a] Department of Mathematics, Southeast University, Nanjing 210096, PR China

[b] Division of Applied Mathematics, Brown University, Providence, RI 02912, USA

ARTICLE INFO

Article history:
Received 5 July 2011
Received in revised form 29 December 2012

Keywords:
Stochastic modeling
Exponential mean-square stability
Two-step Maruyama methods

ABSTRACT

We are concerned with the exponential mean-square stability of two-step Maruyama methods for stochastic differential equations with time delay. We propose a family of schemes and prove that it can maintain the exponential mean-square stability of the linear stochastic delay differential equation for every step size of integral fraction of the delay in the equation. Numerical results for linear and nonlinear equations show that this family of two-step Maruyama methods exhibits better stability than previous two-step Maruyama methods.

1. Introduction

Stochastic delay differential equations (SDDEs) have been increasingly used to model the effects of noise and time delay on various types of complex systems, such as delayed visual feedback systems [1], control problems [2,3] and the dynamics of noisy bi-stable systems with delay [4]. SDDEs are also used in modeling diseases, for example, epidemic diseases [5], neurological diseases [6], etc., and also in finance SDDEs appear in models of stock markets [7].

Some numerical methods and their convergence and stability properties have been established [8–14] recently, but most of them are on one-step methods. Instead of one-step methods we here focus on stochastic multi-step methods for SDDEs, which have been widely studied for solving stochastic ordinary differential equations (SODEs), i.e. with no time delay.

To extend the multi-step methods for SODEs to those for SDDEs is a nontrivial task and these extensions have not been investigated until recently. For a review of multi-step methods for SODEs, we refer to [15,16]. Some more recent studies are as follows. In [17], certain stochastic linear multi-step methods are constructed; and mean-square convergence rates are obtained; and consistency conditions in the mean-square sense are given for two-step Maruyama methods. Ewald and Témam [18] studied the convergence of a stochastic Adams–Bashforth scheme with application to geophysical applications. Adams-type methods for SODEs are also analyzed in [19], where first-order strong convergence conditions are given. For some special SODEs with additive noise, high order multi-step methods have been discussed in [20].

In this paper, we follow [21] and study two-step Maruyama schemes for the scalar equation

$$\begin{aligned} dX(t) &= f\big(t, X(t), X(t-\tau)\big)dt + g\big(t, X(t), X(t-\tau)\big)dW(t), \quad t \in J, \\ X(t) &= \xi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{1.1}$$

where τ is a positive fixed delay, $J = [0, T]$, $W(t)$ is a one-dimensional standard Wiener process and the functions $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We note that [21] is perhaps the only work on multi-step methods for SDDEs, wherein

* Corresponding author at: Department of Mathematics, Southeast University, Nanjing 210096, PR China. Tel.: +86 13905176912.
E-mail addresses: wrcao@seu.edu.cn, seu_cwr@sina.com (W. Cao), Zhongqiang_zhang@brown.edu (Z. Zhang).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2012.12.026

multi-step methods are proposed for m-dimensional systems of Itô SDDEs with d driving Wiener processes and multi-delay, and their properties are studied concerning consistency, numerical stability and convergence.

Instead of working with the general SDDE (1.1), we study the linear test model (1.2), which can shed some insight on the general SDDE (1.1),

$$\begin{aligned} &dX(t) = [aX(t) + bX(t-\tau)]dt + [cX(t) + dX(t-\tau)]dW(t), \quad t \geq 0, \\ &X(t) = \xi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{1.2}$$

where $a, b, c, d \in \mathbb{R}$, τ is a positive fixed delay, $W(t)$ is a one-dimensional standard Wiener process and $\xi(t)$ is a $C([-\tau, 0]; \mathbb{R})$-valued initial segment. In this work, we aim to derive mean-square stable two-step Maruyama methods for the SDDE (1.2).

The paper is organized in the following way. In Section 2 we provide some necessary notations and preliminaries on SDDEs, including some properties of analytical solutions to Eq. (1.2). Also, in this section the two-step Maruyama methods and their convergence properties are introduced. In Section 3 we derive a series of two-step Maruyama methods and prove that the numerical solution is exponentially stable for the exponentially decaying linear SDDE in mean-square sense. Section 4 illustrates the mean-square stability of these two-step Maruyama methods with numerical examples for the test model (1.2) and a nonlinear equation.

2. Notations and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$, which satisfies the usual conditions (increasing and right-continuous; each $\{\mathcal{F}_t\}$, $t \geq 0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$).

Let $W(t), t \geq 0$ in Eq. (1.2) be $\mathcal{F}_t$-adapted and independent of $\mathcal{F}_0$. Assume $\xi(t), t \in [-\tau, 0]$ to be $\mathcal{F}_0$-measurable and right continuous, and $\mathbb{E}\|\xi\|^2 < \infty$. Here $\|\xi\|$ is defined by $\|\xi\| = \sup_{-\tau\leq t\leq 0} |\xi(t)|$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}$. Throughout the paper, Eqs. (1.1) and (1.2) are interpreted in the Itô sense. Under these usual conditions, Eq. (1.2) has a unique strong solution $X(t) : [-\tau, +\infty) \to \mathbb{R}$, which satisfies Eq. (1.2) and $X(t)$ is a measurable, sample-continuous and $\mathcal{F}_t$-adapted process; see [22,23].

Definition 1 ([24]). The trivial solution of Eq. (1.1) is said to be exponentially mean-square stable, if there exists a pair of constants $\lambda > 0$ and $C > 0$, such that, whenever $\mathbb{E}\|\xi\|^2 < \infty$,

$$\mathbb{E}|X(t, \xi)|^2 \leq C\mathbb{E}\|\xi\|^2 e^{-\lambda t}, \quad t \geq 0. \tag{2.1}$$

The inequality (2.1) implies that $\mathbb{E}|X(t)|^2$ goes to 0 exponentially in t as we assume $\mathbb{E}\|\xi\|^2 < \infty$.

Lemma 2 ([25]). *If the condition*

$$a < -|b| - (|c| + |d|)^2 \tag{2.2}$$

holds, then the trivial solution of Eq. (1.2) is exponentially mean-square stable.

Applying the two-step Maruyama methods to Eq. (1.1) leads to the following

$$\sum_{j=-1}^{1} \alpha_j X_{i-j} = h \sum_{j=-1}^{1} \beta_j f(t_{i-j}, X_{i-j}, X_{i-m-j}) + \sum_{j=0}^{1} \gamma_j g(t_{i-j}, X_{i-j}, X_{i-m-j}) \Delta W_{i-j}, \quad i = 2, 3, \ldots, N, \tag{2.3}$$

where $\alpha_j, \beta_j, \gamma_j$, $(j \in \{-1, 0, 1\})$ are parameters; $h > 0$ is the stepsize in time which satisfies $\tau = mh$ for a positive integer m, and $t_n = nh$, $N = T/h$. The increments $\Delta W_i := W(t_{i+1}) - W(t_i)$, are independent $\mathcal{N}(0, h)$-distributed Gaussian random variables. Suppose that X_i is $\mathcal{F}_{t_i}$-measurable at the mesh-point t_i. Then X_i is an approximation to $X(t_i)$, where for $i \leq 0$, X_i are given by the initial function.

Definition 3. The function $u : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be uniform Lipschitz continuous if there exists a positive constant L_u, such that the function u satisfies

$$|u(t, x_1, x_2) - u(t, y_1, y_2)| \leq L_u(|x_1 - y_1| + |x_2 - y_2|) \tag{2.4}$$

for every $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $t \geq 0$,

When there exists a positive constant K, such that

$$|u(t, x, y)| \leq K(1 + x^2 + y^2)^{\frac{1}{2}} \tag{2.5}$$

for $x, y \in \mathbb{R}$ and $t \geq 0$, we say that u satisfies a linear growth condition.

The characteristic polynomial of (2.3) is given by

$$\rho(\lambda) = \alpha_{-1}\lambda^2 + \alpha_0\lambda + \alpha_1.$$

Definition 4. The method (2.3) is said to fulfill Dahlquist's root condition if (i) the roots of $\rho(\lambda)$ lie on or within the unit circle; (ii) the roots on the unit circle are simple.

Here are the consistency and convergence properties of numerical methods (2.3).

Lemma 5 (*[21]*). *Assume that*

- *the coefficients f and g of the SDDE (1.1) are Lipschitz continuous in the sense of (2.4) and have first-order continuous partial derivatives with respect to the first variable and second-order continuous partial derivatives with respect to the second and third variables;*
- *these partial derivatives satisfy the linear growth condition (2.5);*
- *the coefficients of the stochastic linear two-step Maruyama scheme (2.3) satisfy Dahlquist's root condition,*
- *and the consistency conditions*

$$\sum_{j=-1}^{1} \alpha_j = 0, \qquad 2\alpha_{-1} + \alpha_0 = \sum_{j=-1}^{1} \beta_j, \quad \alpha_{-1} = \gamma_0,\ \alpha_{-1} + \alpha_0 = \gamma_1. \tag{2.6}$$

Then the global error of the scheme (2.3) applied to (1.1) satisfies

$$\max_{i=2,\dots,N} \mathbb{E}|X(t_i) - X_i|^2 = O(h^{1/2}).$$

3. Mean-square-stability of the two-step Maruyama methods

In this section, we will derive the schemes in the class of two-step Maruyama methods for Eq. (1.2) and determine a series of two-step Maruyama schemes, which are exponentially mean-square stable, see Theorem 7.

Based on the results in [8,26,27], we give the following definition.

Definition 6. A numerical method is said to be exponentially mean-square stable, if there exist a constant $C > 0$ and $\lambda_h > 0$, such that, whenever $\mathbb{E}\|\xi\|^2 < \infty$, the numerical solution X_n of Eq. (1.2), which has an exponentially mean-square stable trivial solution, at the mesh $t_n = nh,\ n \geq 0$, satisfies

$$\mathbb{E}(X_n)^2 \leq C\mathbb{E}\|\xi\|^2 e^{-\lambda_h t_n}, \quad \text{as } n \to \infty,$$

for fixed step size h under the constraint $h = \tau/m$, where m is a positive integer.

Applying the two-step Maruyama methods (2.3) to Eq. (1.2) gives

$$\sum_{j=-1}^{1} \alpha_j X_{i-j} = h \sum_{j=-1}^{1} \beta_j [aX_{i-j} + bX_{i-m-j}] + \sum_{j=0}^{1} \gamma_j [cX_{i-j} + dX_{i-m-j}]\Delta W_{i-j}, \quad i = 2, 3, \dots; \tag{3.1}$$

for $i \leq 0$, we have $X_i = \xi(t_i)$. For better accuracy, we compute X_1 using the Milstein method with small step size, which has convergence rate $O(h)$ in mean-square sense (see [10]). We also suppose that X_1 is $\mathcal{F}_{t_1}$-measurable at the mesh-point t_1.

By choosing the parameters of the two-step Maruyama method to satisfy the consistency condition (2.6) and

$$\alpha_{-1} = 1, \qquad -1 \leq \alpha_0 < 0, \qquad \beta_0 = \beta_1 = 0, \tag{3.2}$$

then we get

$$\alpha_1 = -1 - \alpha_0, \qquad \beta_{-1} = 2 + \alpha_0, \qquad \gamma_0 = 1, \qquad \gamma_1 = 1 + \alpha_0. \tag{3.3}$$

Thus, we obtain a family of two-step Maruyama schemes from (3.1):

$$\begin{aligned} X_{i+1} + \alpha_0 X_i + (-1 - \alpha_0)X_{i-1} = {} & h(2 + \alpha_0)(aX_{i+1} + bX_{i-m+1}) + (cX_i + dX_{i-m})\Delta W_i \\ & + (1 + \alpha_0)(cX_{i-1} + dX_{i-m-1})\Delta W_{i-1}, \end{aligned} \tag{3.4}$$

where a parameter $-1 \leq \alpha_0 < 0$.

Next we determine the conditions on the parameters for mean square stability. It can be checked that the schemes (3.4) satisfy Dahlquist's root condition and all assumptions in Lemma 5. Thus, we have the following conclusion on mean-square exponential stability.

Theorem 7. *Assume that the condition (2.2) holds. If the parameters of the two-step Maruyama method (3.1) satisfy the restricted conditions (3.2) and (3.3), then the method is exponentially mean-square stable.*

The proof of this theorem needs the following lemma.

Lemma 8. *Under condition (2.2), the numerical solutions $\{X_i,\ i \geq 2\}$ produced by the two-step Maruyama method (3.4) satisfy*

$$\mathbb{E}(X_i X_{i-1} \Delta W_{i-1}) < |c| h \mathbb{E}(X_{i-1}^2) + |d| h \mathbb{E}(|X_{i-1} X_{i-m-1}|), \tag{3.5a}$$

$$\mathbb{E}(X_i X_{i-m-1} \Delta W_{i-1}) < |c| h \mathbb{E}(|X_{i-1} X_{i-m-1}|) + |d| h \mathbb{E}(X_{i-m-1}^2). \tag{3.5b}$$

Proof. Note that $\mathbb{E}(\Delta W_k) = 0$, $\mathbb{E}\big[(\Delta W_k)^2\big] = h$ and X_j is $\mathcal{F}_{t_i}$-measurable and independent of ΔW_i if $j \leq i$. Hence from properties of conditional expectation we can get

$$\mathbb{E}(X_i X_j \Delta W_k) = 0, \quad i, j \leq k, \tag{3.6}$$

$$\mathbb{E}(X_i X_j \Delta W_k \Delta W_{k-1}) = 0, \quad i, j \leq k, \tag{3.7}$$

$$\mathbb{E}(X_i X_j \Delta W_k^2) = h \mathbb{E}(X_i X_j), \quad i, j \leq k. \tag{3.8}$$

Now we prove the inequality (3.5a). From the scheme (3.4) we have

$$\begin{aligned} X_{i+1} = \frac{1}{(1-(2+\alpha_0)ah)} \Big[& (2+\alpha_0) bh X_{i-m+1} + (1+\alpha_0) X_{i-1} - \alpha_0 X_i \\ & + (cX_i + dX_{i-m}) \Delta W_i + (1+\alpha_0)(cX_{i-1} + dX_{i-m-1}) \Delta W_{i-1} \Big] \end{aligned}$$

and then

$$\begin{aligned} X_i = \frac{1}{(1-(2+\alpha_0)ah)} \Big[& (2+\alpha_0) bh X_{i-m} + (1+\alpha_0) X_{i-2} - \alpha_0 X_{i-1} \\ & + (cX_{i-1} + dX_{i-m-1}) \Delta W_{i-1} + (1+\alpha_0)(cX_{i-2} + dX_{i-m-2}) \Delta W_{i-2} \Big]. \end{aligned}$$

Due to the condition (2.2) and $-1 \leq \alpha_0 < 0$, we get $1-(2+\alpha_0)ah > 1$. Using (3.6)–(3.8), it holds that

$$\begin{aligned} \mathbb{E}(X_i X_{i-1} \Delta W_{i-1}) &= \frac{1}{(1-(2+\alpha_0)ah)} \Big[(2+\alpha_0) bh \mathbb{E}(X_{i-m} X_{i-1} \Delta W_{i-1}) + (1+\alpha_0) \mathbb{E}(X_{i-2} X_{i-1} \Delta W_{i-1}) \\ &\quad - \alpha_0 \mathbb{E}(X_{i-1}^2 \Delta W_{i-1}) + c \mathbb{E}(X_{i-1}^2 \Delta W_{i-1}^2) + d \mathbb{E}(X_{i-m-1} X_{i-1} \Delta W_{i-1}^2) \\ &\quad + (1+\alpha_0) c \mathbb{E}(X_{i-2} \Delta W_{i-2} X_{i-1} \Delta W_{i-1}) + (1+\alpha_0) d \mathbb{E}(X_{i-m-2} \Delta W_{i-2} X_{i-1} \Delta W_{i-1}) \Big] \\ &= \frac{1}{(1-(2+\alpha_0)ah)} \Big(ch \mathbb{E}(X_{i-1}^2) + dh \mathbb{E}(X_{i-m-1} X_{i-1}) \Big) \\ &< |c| h \mathbb{E}(X_{i-1}^2) + |d| h \mathbb{E}(|X_{i-m-1} X_{i-1}|). \end{aligned}$$

Inequality (3.5b) can be proved in the same way. This proves the lemma. □

Proof of Theorem 7. The explicit form of the scheme (3.4) is

$$\begin{aligned} \Big(1-(2+\alpha_0)ah\Big) X_{i+1} &= (2+\alpha_0) bh X_{i-m+1} + \Big((1+\alpha_0) X_{i-1} - \alpha_0 X_i\Big) + (cX_i + dX_{i-m}) \Delta W_i \\ &\quad + (1+\alpha_0)(cX_{i-1} + dX_{i-m-1}) \Delta W_{i-1}. \end{aligned}$$

We square both sides of the last difference equation to obtain

$$\begin{aligned} (1-(2+\alpha_0)ah)^2 X_{i+1}^2 &= (2+\alpha_0)^2 b^2 h^2 X_{i-m+1}^2 + \Big((1+\alpha_0) X_{i-1} - \alpha_0 X_i\Big)^2 + (cX_i + dX_{i-m})^2 \Delta W_i^2 \\ &\quad + (1+\alpha_0)^2 (cX_{i-1} + dX_{i-m-1})^2 \Delta W_{i-1}^2 + 2(2+\alpha_0) bh X_{i-m+1} \Big((1+\alpha_0) X_{i-1} - \alpha_0 X_i\Big) \\ &\quad + 2(2+\alpha_0) bh X_{i-m+1} (cX_i + dX_{i-m}) \Delta W_i + 2(2+\alpha_0) bh X_{i-m+1} (1+\alpha_0)(cX_{i-1} \\ &\quad + dX_{i-m-1}) \Delta W_{i-1} + 2\Big((1+\alpha_0) X_{i-1} - \alpha_0 X_i\Big)(cX_i + dX_{i-m}) \Delta W_i \\ &\quad + 2\Big((1+\alpha_0) X_{i-1} - \alpha_0 X_i\Big)(1+\alpha_0)(cX_{i-1} + dX_{i-m-1}) \Delta W_{i-1} \\ &\quad + 2(1+\alpha_0)(cX_i + dX_{i-m})(cX_{i-1} + dX_{i-m-1}) \Delta W_i \Delta W_{i-1}. \end{aligned}$$

Then

$$\begin{aligned}(1-(2+\alpha_0)ah)^2X_{i+1}^2 = {} & \alpha_0^2X_i^2 + c^2\Delta W_i^2X_i^2 + (1+\alpha_0)^2X_{i-1}^2 + (1+\alpha_0)^2c^2\Delta W_{i-1}^2X_{i-1}^2 + (2+\alpha_0)^2h^2b^2X_{i-m+1}^2 \\ & + d^2\Delta W_i^2X_{i-m}^2 + (1+\alpha_0)^2d^2\Delta W_{i-1}^2X_{i-m-1}^2 - 2\alpha_0(1+\alpha_0)X_iX_{i-1} \\ & - 2\alpha_0(1+\alpha_0)c\Delta W_{i-1}X_iX_{i-1} - 2\alpha_0(2+\alpha_0)bhX_iX_{i-m+1} \\ & - 2\alpha_0(1+\alpha_0)d\Delta W_{i-1}X_iX_{i-m-1} + 2(1+\alpha_0)(2+\alpha_0)bhX_{i-1}X_{i-m+1} \\ & + 2cd\Delta W_i^2X_iX_{i-m} + 2(1+\alpha_0)^2cd\Delta W_{i-1}^2X_{i-1}X_{i-m-1} + O_i(\Delta W_i, \Delta W_{i-1}),\end{aligned} \tag{3.9}$$

where we define

$$\begin{aligned}O_i(\Delta W_i, \Delta W_{i-1}) = {} & 2(1+\alpha_0)^2(cX_{i-1}+dX_{i-m-1})X_{i-1}\Delta W_{i-1} + 2(2+\alpha_0)bhX_{i-m+1}(cX_i+dX_{i-m})\Delta W_i \\ & + 2(2+\alpha_0)bhX_{i-m+1}(1+\alpha_0)(cX_{i-1}+dX_{i-m-1})\Delta W_{i-1} + 2\Big((1+\alpha_0)X_{i-1}-\alpha_0X_i\Big) \\ & \times (cX_i+dX_{i-m})\Delta W_i + 2(1+\alpha_0)(cX_i+dX_{i-m})(cX_{i-1}+dX_{i-m-1})\Delta W_i\Delta W_{i-1}.\end{aligned}$$

By the linearity of expectation and (3.6)–(3.7) in Lemma 8, we get

$$\mathbb{E}\left(O_i(\Delta W_i, \Delta W_{i-1})\right) = 0.$$

Let $Y_i = \mathbb{E}(X_i^2),\ i = 0, 1, 2, \ldots$. Taking the expectation over both sides of (3.9), and using the inequality $2ab \le a^2 + b^2$ and (3.5a), (3.5b) and (3.8) in Lemma 8, it follows that

$$P_0Y_{i+1} \le P_1Y_i + P_2Y_{i-1} + P_3Y_{i-m+1} + P_4Y_{i-m} + P_5Y_{i-m-1}, \quad i = 2, 3, \ldots, \tag{3.10}$$

where

$$\begin{aligned}P_0 &= (1-(2+\alpha_0)ah)^2, \\ P_1 &= \alpha_0^2 + c^2h - \alpha_0(1+\alpha_0) - \alpha_0(2+\alpha_0)|b|h + |cd|h, \\ P_2 &= (1+\alpha_0)^2c^2h + (1+\alpha_0)^2 - \alpha_0(1+\alpha_0) - 2\alpha_0(1+\alpha_0)c^2h \\ &\quad - 2\alpha_0(1+\alpha_0)|cd|h + (1+\alpha_0)(2+\alpha_0)|b|h + (1+\alpha_0)^2|cd|h, \\ P_3 &= (2+\alpha_0)^2b^2h^2 - \alpha_0(2+\alpha_0)|b|h + (1+\alpha_0)(2+\alpha_0)|b|h, \\ P_4 &= d^2h + |cd|h, \\ P_5 &= (1+\alpha_0)^2(d^2+|cd|)h - 2\alpha_0(1+\alpha_0)d^2h - 2\alpha_0(1+\alpha_0)|cd|h.\end{aligned}$$

Let $P = P(a, b, c, d, \alpha_0, h) = \big(P_1 + P_2 + P_3 + P_4 + P_5\big)/P_0$. It is obvious that

$$Y_{i+1} \le P\max\big\{Y_i, Y_{i-1}, Y_{i-m+1}, Y_{i-m}, Y_{i-m-1}\big\}, \quad i = 2, 3, \ldots. \tag{3.11}$$

We now claim that, for any stepsize $h = \tau/m$,

$$Y_{i+1} \le \max\left\{P^{i+1}, P^i, \ldots, P^{\left[\frac{i-m-1}{m+2}\right]+1}\right\}\mathbb{E}\|\xi\|^2. \tag{3.12}$$

Thus, if $P < 1$, then we can get $\lim_{i\to\infty} Y_i = 0$, as $\mathbb{E}\|\xi\|^2 < \infty$. In fact, we use recurrence method to inequality (3.11) and get

$$\begin{aligned}Y_{i+1} &\le P\max\{Y_i, Y_{i-1}, Y_{i-m+1}, Y_{i-m}, Y_{i-m-1}\} \\ &\le P^2\max\{Y_{i-1}, Y_{i-2}, \ldots, Y_{i-2m-3}\} \\ &\quad \cdots \\ &\le P^{\left[\frac{i-m-1}{m+2}\right]+1}\max\left\{Y_{i-\left[\frac{i-m-1}{m+2}\right]}, \ldots, Y_1, \mathbb{E}\|\xi\|^2\right\} \\ &\le P^{\left[\frac{i-m-1}{m+2}\right]+2}\max\left\{Y_{i-\left[\frac{i-m-1}{m+2}\right]-1}, \ldots, \mathbb{E}\|\xi\|^2, \frac{1}{P}\mathbb{E}\|\xi\|^2\right\} \\ &\quad \cdots \\ &\le P^{i+1}\max\left\{\mathbb{E}\|\xi\|^2, \frac{1}{P}\mathbb{E}\|\xi\|^2, \ldots, \frac{1}{P^{i-\left[\frac{i-m-1}{m+2}\right]-1}}\mathbb{E}\|\xi\|^2\right\} \\ &\le \max\left\{P^{i+1}, P^i, \ldots, P^{\left[\frac{i-m-1}{m+2}\right]+1}\right\}\mathbb{E}\|\xi\|^2.\end{aligned} \tag{3.13}$$

Table 1
Parameters for different two-step Maruyama schemes.

Scheme	α_0	β_{-1}	β_0
Two-step method 1 (TS1)	$-1/2$	$1/4$	$5/4$
Two-step method 2 (TS2)	$-3/2$	$1/2$	1
Two-step method 3 (TS3)	$\mathbf{-1/2}$	$\mathbf{3/2}$	$\mathbf{0}$
Two-step method 4 (TS4)	$-1/3$	$1/3$	$4/3$
Two-step method 5 (TS5)	$\mathbf{-2/3}$	$\mathbf{4/3}$	$\mathbf{0}$
Two-step method 6 (TS6)	$-4/3$	0	$2/3$

The third inequality is obtained because the index of Y_{i-m-1} in (3.11) keeps reducing by $m-2$ after each recurrence process. The fifth inequality holds since the index of Y_i in (3.11) declines by 1 after each recurrence. For the reason that $\mathbb{E}\|\xi\|^2$ is eliminated from the recurrence process, we divide it by P for each time and eventually we get $\frac{1}{P^{i-[\frac{i-m-1}{m+2}]-1}}\mathbb{E}\|\xi\|^2$.

In the process of iteration, Y_{i-m-1} $(i \geq m+1)$ will be the first term down to $Y_0 = \mathbb{E}\|\xi\|^2$ and it calls for its previous $[\frac{i-m-1}{m+2}]+1$ steps, where $[z]$ means the largest integer no more than a real number z. This proves the claim (3.12).

It is essential to verify that $P < 1$, i.e. $P_1+P_2+P_3+P_4+P_5 < P_0$. Recall that $-1 \leq \alpha_0 < 0$ and thus $2-\alpha_0^2 \leq 2(2+\alpha_0)$, then from (3.10), we have

$$\begin{aligned}
P_1+P_2+P_3+P_4+P_5-P_0 &= 1+\Big(2(2+\alpha_0)|b|+(2-\alpha_0^2)(|c|+|d|)^2\Big)h+(2+\alpha_0)^2b^2h^2-(1-(2+\alpha_0)ah)^2 \\
&= (2+\alpha_0)^2(b^2-a^2)h^2+2\left((2+\alpha_0)a+(2+\alpha_0)b+\frac{1}{2}(2-\alpha_0^2)(|c|+|d|)^2\right)h \\
&\leq (2+\alpha_0)^2(b^2-a^2)h^2+2(2+\alpha_0)\Big(a+|b|+(|c|+|d|)^2\Big)h. \qquad (3.14)
\end{aligned}$$

Based on condition (2.2), we obtain that $P_1+P_2+P_3+P_4+P_5-P_0 < 0$ holds for each stepsize $h = \tau/m$.

Due to $P < 1$, we obtain $Y_{i+1} \leq P^{[\frac{i-m-1}{m+2}]+1}\mathbb{E}\|\xi\|^2$ from (3.13) and thus

$$Y_{i+1} \leq P^{\left[\frac{i-m-1}{m+2}\right]+1}\mathbb{E}\|\xi\|^2 \leq \left(P^{\frac{1}{m+2}}\right)^{i+1}\mathbb{E}\|\xi\|^2 = e^{-\lambda_h t_{i+1}}\mathbb{E}\|\xi\|^2,$$

where $\lambda_h = -\frac{\ln P}{(m+2)h} > 0$. The proof ends. □

4. Numerical examples

In all our numerical examples,

$$\mathbb{E}(X_n^2) = \frac{1}{2000}\sum_{i=1}^{2000}|X_n(\omega_i)|^2,$$

are the sampled average over 2000 trajectories in Matlab.

Table 1 lists a number of two-step Maruyama schemes with different parameters under test here. Note that only the schemes in bold (TS3 and TS5) satisfy the required conditions in Theorem 7 and hence are exponentially mean-square stable. However, the other Maruyama schemes TS1, TS2, TS4, TS6 do not satisfy the conditions in Theorem 7 and thus may be only conditionally stable and even not stable as we show later on.

Example 1. We consider the linear test model

$$\begin{aligned}
&dX(t) = [aX(t)+bX(t-\tau)]dt+[cX(t)+dX(t-\tau)]dW(t), \quad t \geq 0, \\
&X(t) = t+\tau, \quad t \in [-\tau, 0]
\end{aligned} \qquad (4.1)$$

to illustrate the mean-square stability of the two-step Maruyama schemes in Table 1.

We choose the parameters as $a = -4$, $b = 2$, $c = 0.5$, $d = 0.5$ and $\tau = 1$, which ensures that the exact solution of the Eq. (4.1) is mean-square stable by Lemma 2. From Fig. 1, we observe that TS1 is not mean-square stable for both large stepsize $h = 1/4$ and small stepsize $h = 1/64$. From Figs. 2 and 3, we see that TS3 and TS5 are mean-square stable even for a large stepsize $h = 1/4$; Fig. 2 illustrates that TS4 is conditionally mean-square stable; TS4 is mean square stable if the stepsize h is small enough (like $h = 1/64$ here).

In Fig. 3, we fix the stepsize $h = 1/8$ and show that to a great extent the mean square stability of the implicit scheme TS3 is better than the explicit two-step scheme TS6. If we test TS6 for a rather long time interval, then the numerical solution

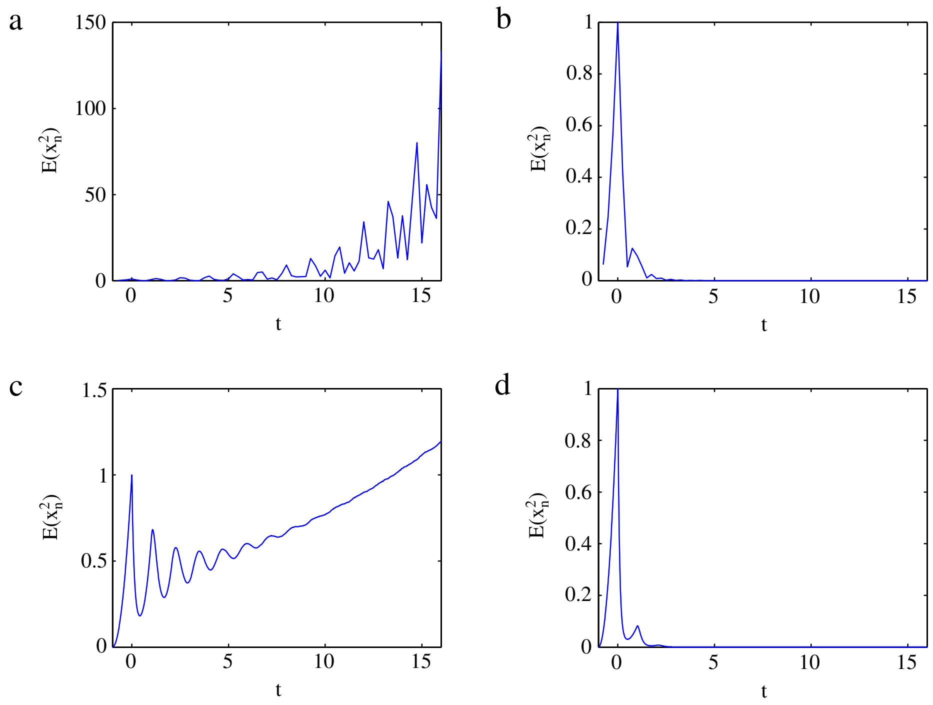

Fig. 1. Simulations with TS1 and TS3. (a): TS1, $h = 1/4$; (b): TS3, $h = 1/4$; (c): TS1, $h = 1/64$; (d): TS3, $h = 1/64$.

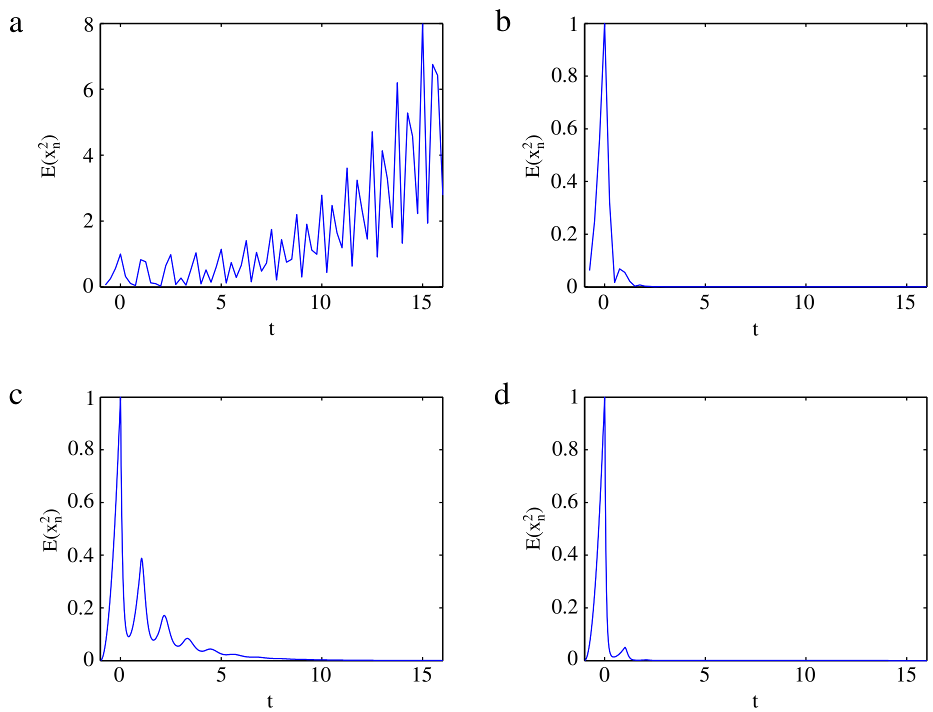

Fig. 2. Simulations with TS4 and TS5. (a): TS4, $h = 1/4$; (b): TS5, $h = 1/4$; (c): TS4, $h = 1/64$; (d): TS5, $h = 1/64$.

will oscillate and will finally diverge. From Figs. 1–3, we observe that the numerical solution from TS6 blows up earlier than any other unstable implicit methods shown in Figs. 1 and 2. On the other hand, numerical solutions of both TS3 and TS5 converge . to zero very fast.

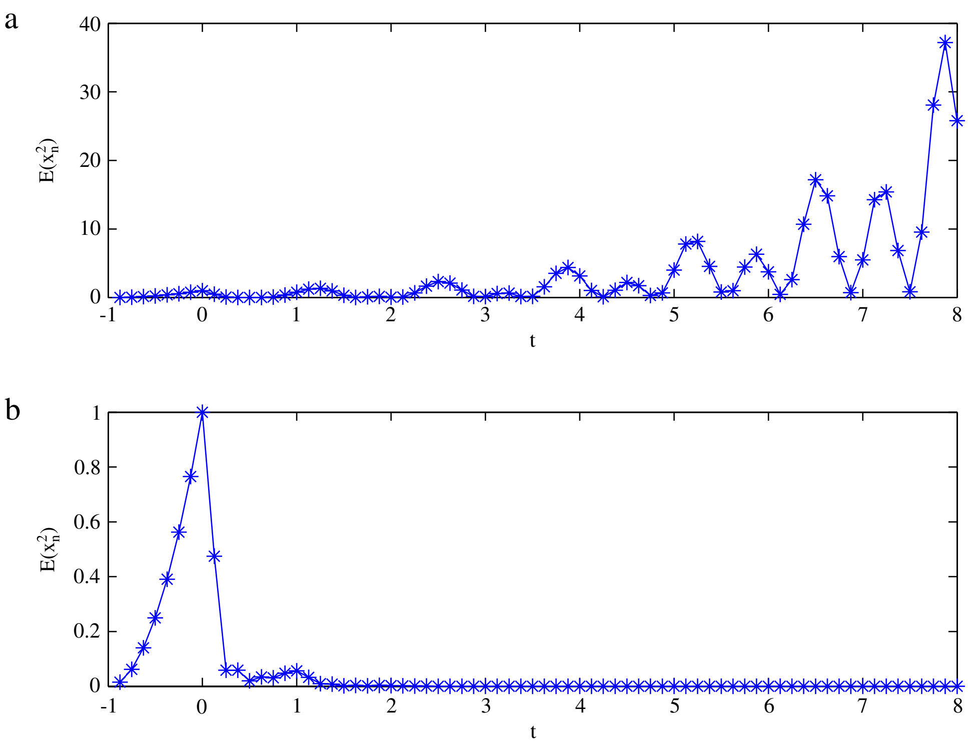

Fig. 3. Simulations with TS3 and the explicit method TS6. (a): TS6; (b): TS3. $h = 1/8$.

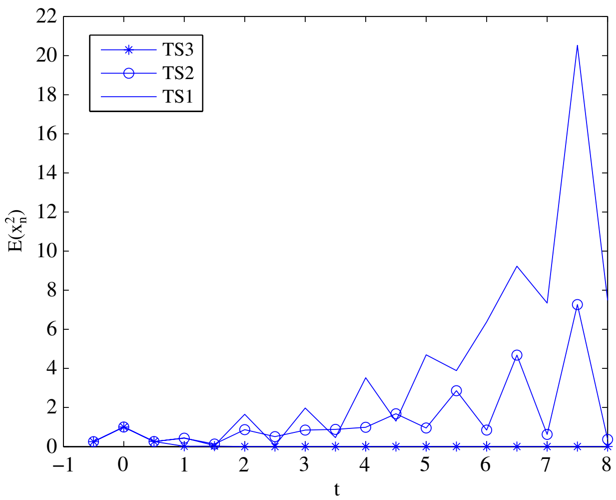

Fig. 4. Simulations with TS1, TS2 and TS3 for fixed stepsize $h = 1/2$.

In Fig. 4 we choose different parameters for the Eq. (4.1): $a = -3, b = 1, c = 0.5, d = 0.5$ and $\tau = 1$ when (4.1) is mean square stable. Here we use a very large stepsize $h = 1/2$ for TS1, TS2 and TS3. The results show that the scheme TS3 maintains the mean-square stability even with large stepsize h.

Example 2. We test the proposed two-step Maruyama methods for the following nonlinear SDDE (Example 5.2.1, [28]):

$$\begin{aligned} dX(t) &= -\frac{a}{1+t}X(t) + \frac{b}{1+t}X(t)\sin(X(t-\tau))dW(t), \quad t \geq 0, \\ X(t) &= t + \tau, \quad t \in [-\tau, 0]. \end{aligned} \tag{4.2}$$

The solution of Eq. (4.2) is mean square stable if $2a - 1 \geq b^2$ and $b \neq 0$ [28].

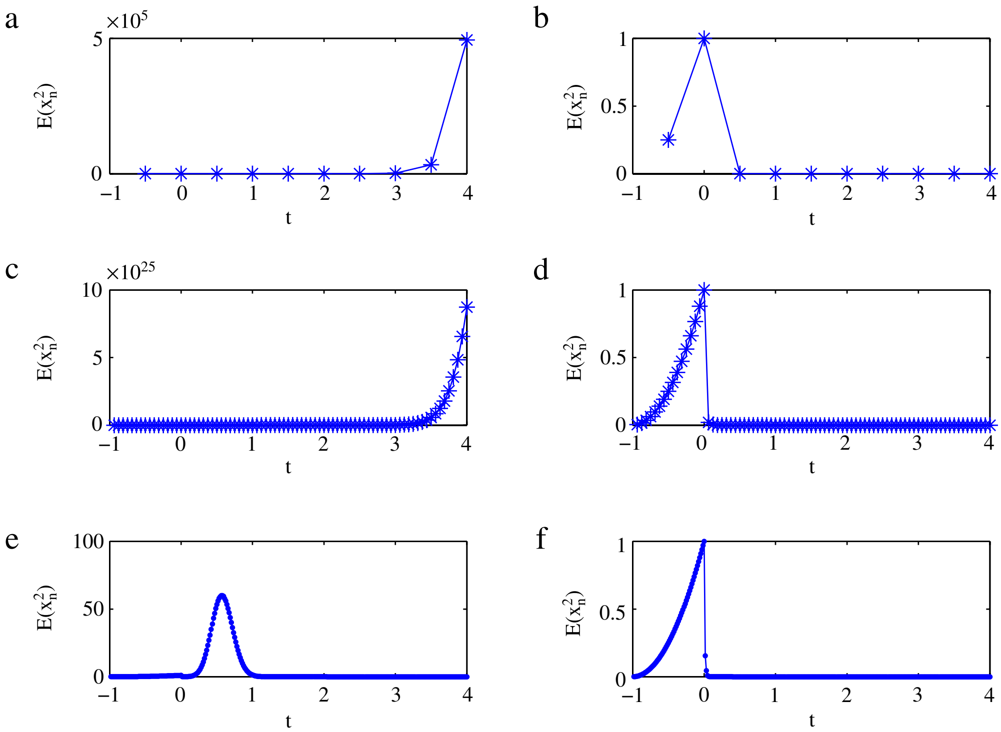

Fig. 5. Simulations with TS1 and TS3. (a): TS1, $h = 1/2$; (b): TS3, $h = 1/2$; (c): TS1, $h = 1/16$; (d): TS3, $h = 1/16$; (e): TS1, $h = 1/64$; (f): TS3, $h = 1/64$.

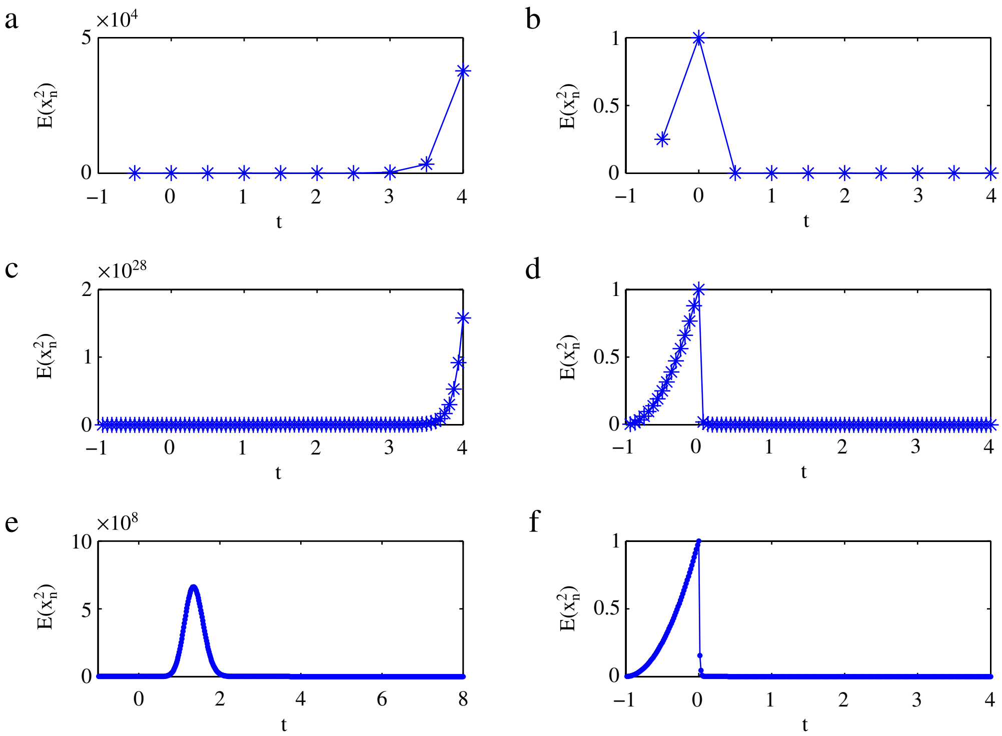

Fig. 6. Simulations with TS4 and TS5. (a): TS4, $h = 1/2$; (b): TS5, $h = 1/2$; (c): TS4, $h = 1/16$; (d): TS5, $h = 1/16$; (e): TS4, $h = 1/64$; (f): TS5, $h = 1/64$.

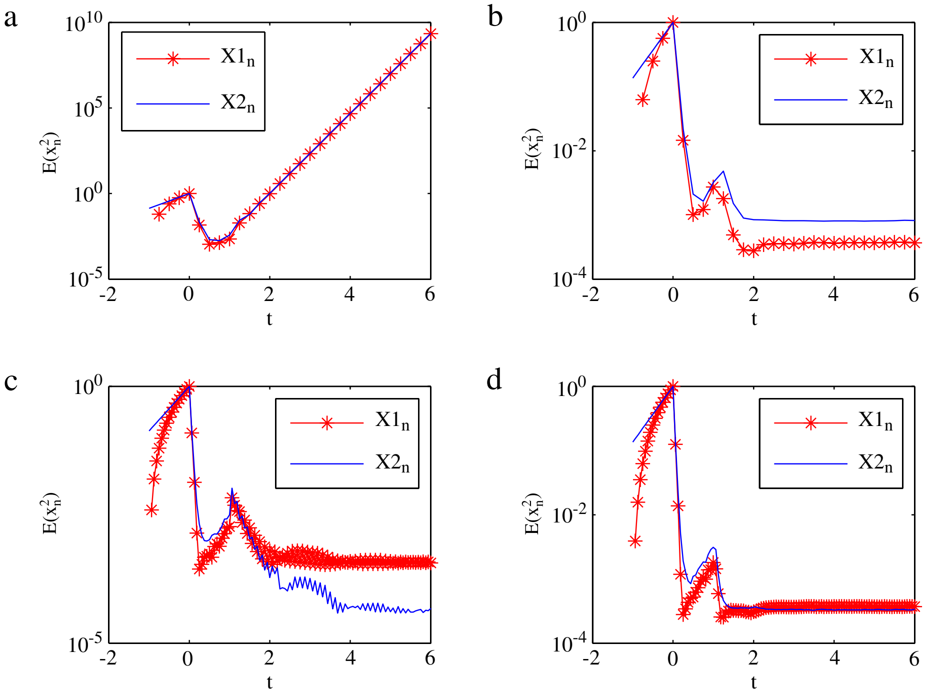

Fig. 7. Simulations with TS2 and TS5. (a): TS2, $h = 1/4$; (b): TS5, $h = 1/4$; (c): TS2, $h = 1/16$; (d): TS5, $h = 1/16$.

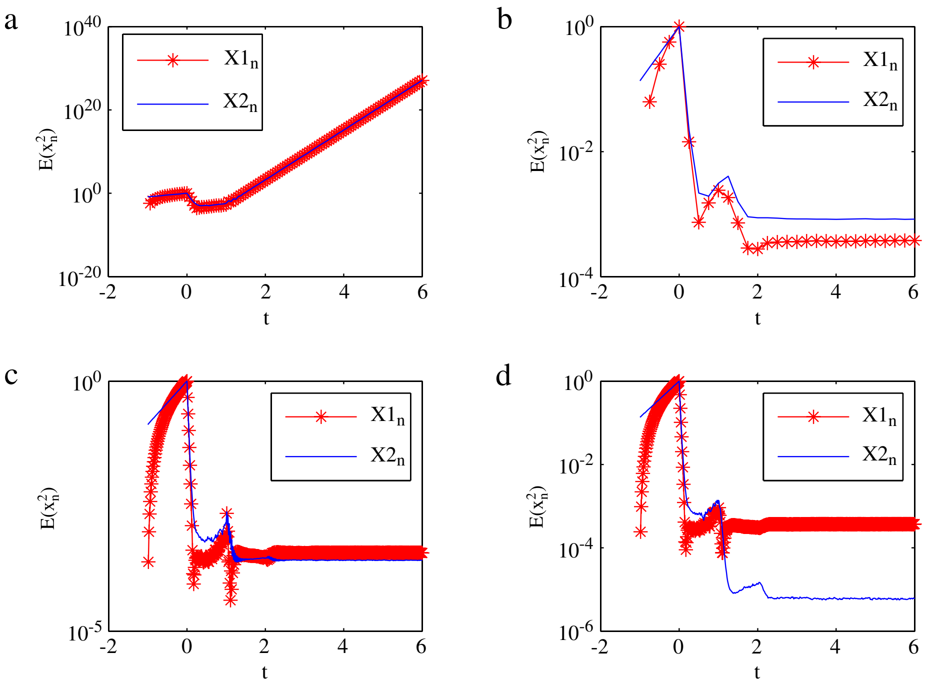

Fig. 8. Simulations with TS3 and TS4. (a): TS4, $h = 1/16$; (b): TS3, $h = 1/4$; (c): TS4, $h = 1/64$; (d): TS3, $h = 1/64$.

We take $a = 100$, $b = 10$ and $\tau = 1$.

It is shown from Figs. 5 and 6 that the schemes TS3 and TS5 maintain their mean-square stability for nonlinear SDDE (4.2); TS1 and TS4 are conditionally mean-square stable. In some range of stepsize h, smaller h leads to a greater instability, comparing to (a), (c) in Fig. 5 and (a), (c) in Fig. 6.

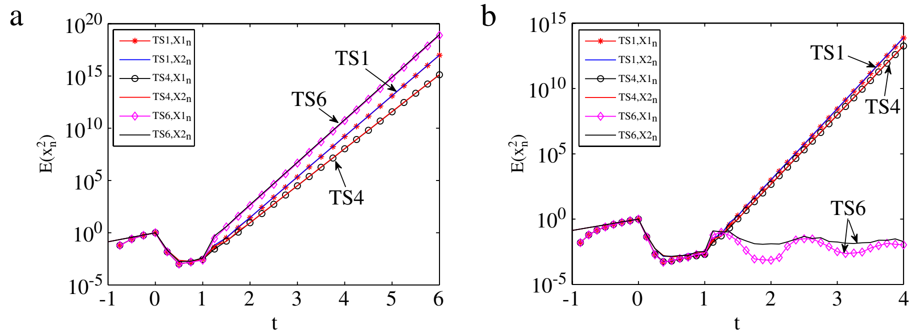

Fig. 9. Simulations with TS1, TS4 and TS6. (a): $h = 1/4$; (b): $h = 1/8$.

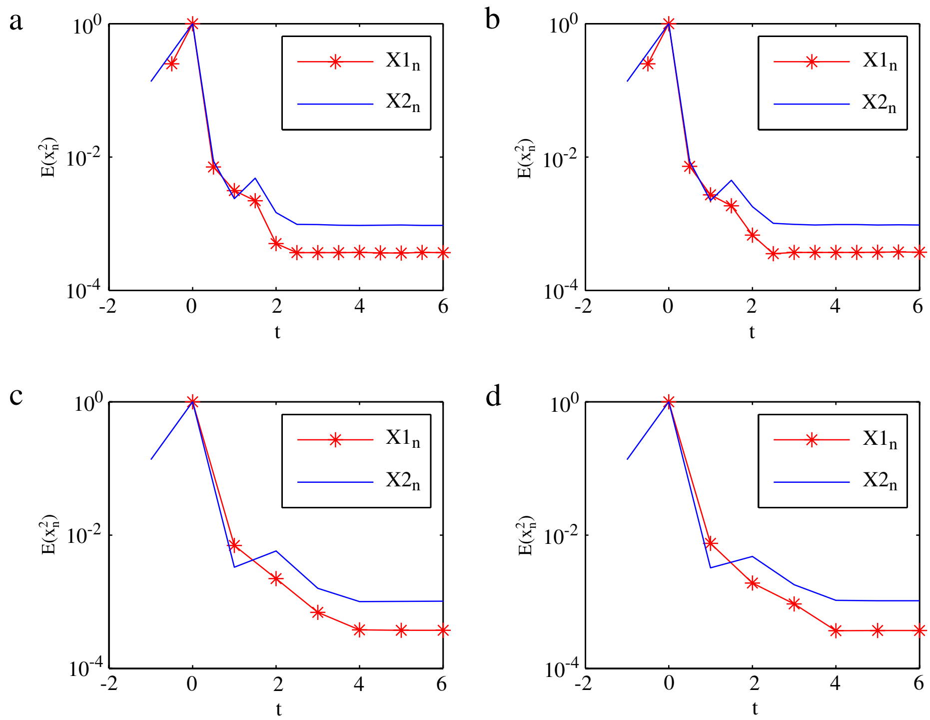

Fig. 10. Simulations with TS3 and TS5. (a): TS3, $h = 1/2$; (b): TS5, $h = 1/2$; (c): TS3, $h = 1$; (d): TS5, $h = 1$.

Example 3. We test the proposed two-step Maruyama methods for the following nonlinear stochastic delay differential 2-dimensional system:

$$\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \left[A \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} + B \begin{pmatrix} \sin(X_1(t-\tau)) \\ \cos(X_2(t-\tau)) \end{pmatrix} \right] dt + C \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}, \tag{4.3}$$

where

$$A = \begin{pmatrix} -28 & 0 \\ 0 & -30 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -1/2 \\ 1/4 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 3/2 \\ 5/2 & -1/2 \end{pmatrix}.$$

When $t \in [-\tau, 0]$, $X_1(t) = t + \tau$ and $X_2(t) = e^t$. Based on Corollary 2.2 [24], we know that solutions of system (4.3) are exponentially mean-square stable.

From Fig. 7, we observe that the solutions obtained by scheme TS2 blow up for $h = 1/4$ and are exponentially mean-square stable when $h = 1/16$ ((a) and (c)) and the solutions obtained from exponentially mean-square stable scheme TS5 are mean-square stable for both $h = 1/4$ and $h = 1/16$, see Fig. 7(b) and (d). It is shown from Fig. 8 that TS3 performs good exponentially mean-square stability. Meanwhile, TS4 is conditionally mean-square stable, and the smaller step size $h = 1/64$ is needed (compare to TS2).

We compare solutions obtained by the schemes TS1, TS4 and TS6 in Fig. 9. Fig. 9(a) shows that neither of them are mean-square stable for $h = 1/4$, and the solution from the explicit scheme TS6 blows up faster than that from implicit scheme TS1 and TS4. On the other hand, the explicit scheme TS6 performs better than the scheme TS1 and TS4 when $h = 1/8$ as we can see from Fig. 9(b) that TS6 is mean-square stable for $h = 1/8$ but TS1 and TS4 are not. Fig. 9 indicates that the explicit scheme TS6 requires less restricted time step size h for mean-square stability than the implicit scheme TS1 and TS4 do.

In Fig. 10, we test the proposed schemes TS3 and TS5 with very large step size $h = 1/2$ and $h = 1$, the schemes TS3 and TS5 are maintaining their exponential stability in mean-square sense for nonlinear stochastic delay differential system (4.3).

5. Conclusion

We have proposed a family of exponentially mean-square stable two-step Maruyama schemes. It has been proved that the proposed schemes can maintain the exponential mean-square stability of the linear SDDE for every step size of integral fraction of the delay in the equation. Numerical examples show that this family of numerical methods exhibits exponential mean-square stability for both linear and some particular nonlinear SDDE and 2-dimensional SDDEs. The numerical results suggest that our proposed scheme can be adopted for general nonlinear SDDEs, but further numerical studies are required.

Acknowledgments

The authors thank the referees and Dr. Taketomo Mitsui for their helpful suggestions for improving the paper. The first author also would like to thank Professor George Em Karniadakis for his hospitality when she was visiting Division of Applied Mathematics at Brown University.

First author's work was supported by the NSF of China (No. 10901036).

Second author's work was supported by AIRFORCE MURI.

References

[1] A. Beuter, K. Vasilakos, Effects of noise on a delayed visual feedback system, J. Theoret. Biol. 165 (1993) 389–407.
[2] M. Grigoriu, Control of time delay linear systems with Gaussian white noise, Probab. Eng. Mech. 12 (1997) 89–96.
[3] M.D. Paola, A. Pirrotta, Time delay induced effects on control of linear systems under random excitation, Probab. Eng. Mech. 16 (2001) 43–51.
[4] L.S. Tsimring, A. Pikovsky, Noise-induced dynamics in bistable systems with delay, Phys. Rev. Lett. 87 (2001) 250602 [4 pages].
[5] E. Beretta, V.B. Kolmanovskii, L. Shaikhet, Stability of epidemic model with time delays influenced by stochastic perturbations, Math. Comput. Simul. 45 (1998) 269–277.
[6] A. Beuter, J. Belair, Feedback and delays in neurological diseases: a modeling study using dynamical systems, Bull. Math. Biol. 55 (1993) 525–541.
[7] D.G. Hobson, L.C.G. Rogers, Complete models with stochastic balatility, Math. Finance 8 (1998) 27–48.
[8] C.T.H. Baker, E. Buckwar, Exponential stability in p-th mean of solutions, and of convergent Euler-type solutions, of stochastic delay differential equations, J. Comput. Appl. Math. 184 (2005) 404–427.
[9] Z. Fan, Waveform relaxation method for stochastic differential equations with constant delay, Appl. Numer. Math. 61 (2011) 229–240.
[10] Y. Hu, S.A. Mohammed, F. Yan, Discrete-time approximations of stochastic delay equations: the Milstein scheme, Ann. Probab. 32 (2004) 265–314.
[11] R. Li, Convergence and stability of numerical solutions to SDDEs with Markovian switching, Appl. Math. Comput. 175 (2006) 1080–1091.
[12] M. Liu, W. Cao, Z. Fan, Convergence and stability of the semi-implicit Euler method for a linear stochastic differential delay equation, J. Comput. Appl. Math. 170 (2004) 255–268.
[13] X. Mao, Numerical solutions of stochastic differential delay equations under the generalized Khasminskii-type conditions, Appl. Math. Comput. 217 (2011) 5512–5524.
[14] S. Zhou, F. Wu, Convergence of numerical solutions to neutral stochastic delay differential equations with Markovian switching, J. Comput. Appl. Math. 229 (2009) 85–96.
[15] P.E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, Springer, Berlin, 1992.
[16] G.N. Milstein, Numerical Integration of Stochastic Differential Equations, Kluwer Academic Publishers Group, Dordrecht, 1995.
[17] E. Buckwar, R. Winkler, Multi-step methods for SDEs and their application to problems with small noise, SIAM J. Numer. Anal. 44 (2006) 779–803.
[18] B.D. Ewald, R. Témam, Numerical analysis of stochastic schemes in geophysics, SIAM J. Numer. Anal. 42 (2005) 2257–2276.
[19] L. Brugnano, K. Burrage, P.M. Burrage, Adams-type methods for the numerical solution of stochastic ordinary differential equations, BIT 40 (2000) 451–470.
[20] G. Denk, S. Schäffler, Adams methods for the efficient solution of stochastic differential equations with additive noise, Computing 59 (1996) 153–161.
[21] E. Buckwar, R. Winkler, Multi-step Maruyama methods for stochastic delay differential equations, Stoch. Anal. Appl. 25 (2007) 933–959.
[22] X. Mao, Stochastic Differential Equations and Applications, Horwood, 1997.
[23] S.E.A. Mohammed, Stochastic Functional Differential Equations, in: Research Notes in Mathematics, vol. 99, Pitman, London, 1984.
[24] X. Mao, Exponential Stability of Stochastic Differential Equations, Marcel Dekker, Inc., New York, 1994.
[25] X. Mao, Razumikhin-type theorems on exponential stability of stochastic functional differential equations, Stoch. Proc. Appl. 65 (1996) 233–250.
[26] V.K. Barwell, Special stability problems for functional equations, BIT 15 (1975) 130–135.
[27] L. Shaikhet, Lyapunov Functionals and Stability of Stochastic Difference Equations, Springer, London, Dordrecht, Heidelberg, New York, 2011.
[28] X. Mao, Stability of Stochastic Differential Equations with Respect to Semimartingales, Longman Scientific & Technical, 1991.