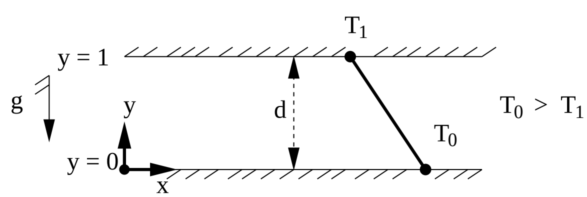
Chapter 9

Examples/Applications

Numerical Simulation of the Rayleigh-Bernard Problem

In this example we consider thermal convection in a thin layer of fluid heated from below. We want to analyze numerically its stability characteristics.



Rayleigh (1916) formulated the theory of convective instability of a layer of fluid between horizontal plates motivated by the experiments of Bernard (1900).

The governing equations

Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_j)}{\partial x_j} = 0$$

Navier-Stokes:

$$\rho \frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} - g\rho\delta_{i2}$$

where

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)$$

Energy:

$$\rho\frac{DE}{Dt} = \frac{\partial}{\partial x_j}\left(k\frac{\partial \theta}{\partial x_j}\right) - p\frac{\partial u_j}{\partial x_j} + \Phi$$

where $E = c_v\theta$ (perfect gas) or $E = c\theta$ (liquid) and the rate of viscous dissipation per unit volume is:

$$\Phi = \frac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 - \frac{2}{3}\mu\left(\frac{\partial u_k}{\partial x_k}\right)^2$$

The *Boussinesq approximation* is based on the assumption that in certain flows where the temperature varies little, the corresponding density varies little but the motion is driven due to buoyancy forces, i.e.,

$$\rho = \rho_0[1 - \beta(\theta - \theta_0)]$$

where $\beta(K^{-1})$ is the coefficient of thermal expansion. For a perfect gas $\beta = \frac{1}{T_0} \approx 3 \times 10^{-3}K^{-1}$. For a liquid $\beta \approx 5 \times 10^{-4}K^{-1}$. Therefore for $\Delta\theta = \Delta T = \theta_0 - \theta < 10K$, then

$$\frac{\rho - \rho_0}{\rho_0} = \beta(\theta_0 - \theta) \ll 1.$$

However, the term $g(\rho - \rho_0) \sim \rho\frac{Du}{Dt}$ and thus it cannot be neglected in the momentum equation. Also, for most fluids

$$\frac{1}{\mu}\frac{d\mu}{d\theta} < \beta$$

etc. so all other physical properties are taken at the reference temperature θ_0. The Boussinesq equations are then:
Continuity:

$$\frac{\partial u_j}{\partial x_j} = 0$$

since we neglect

$$\frac{1}{\rho}\frac{d\rho}{dt} \sim \mathcal{O}(\beta)$$

and thus the stress tensor is like the incompressible one, i.e.,

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j}\right)$$

then Navier-Stokes:

$$\frac{Du_i}{Dt} = -\frac{\partial}{\partial x_i}\left(\frac{p}{\rho_0} + gz\right) - \beta g(\theta_0 - \theta)\delta_{i2} + \nu\nabla^2 u_i$$

The energy equation is simplified similarly. First, the dissipation term is small compared to the convective term, i.e.,

$$\frac{\Phi}{\rho\frac{D(c\theta)}{Dt}} \sim \frac{\mu\left(\frac{V}{d}\right)^2}{\rho_0 c V\frac{\Delta\theta}{d}} = \frac{\nu}{c}\frac{V}{(\theta_0 - \theta_1)d}$$

For liquids, $\frac{\nu}{c} \sim 10^{-9}$; gases $\frac{\nu}{c} \sim 10^{-8}$. Also $\frac{V}{d\Delta\theta}$ is finite (unless we have $d \sim \mathcal{O}(\text{micron})$). Therefore we can neglect the dissipation term. The heating term due to compression is:

$$\begin{aligned} -p\frac{\partial u_j}{\partial x_j} &= \frac{p}{\rho}\frac{D\rho}{Dt} \;\; \text{(continuity)} \\ &= \beta p\frac{D\theta}{Dt} \;\; \text{(Boussinesq)} \end{aligned}$$

For a perfect gas:

$$\begin{aligned} p &= (c_p - c_v)\rho\theta \\ \beta &= \frac{1}{\theta} \end{aligned}$$

and therefore:

$$\rho\frac{DE}{Dt} + p\frac{\partial u_i}{\partial x_j} \cong c_p\rho\frac{D\theta}{Dt}$$

so it is not negligible for gases. For typical liquids it is because the heat transfer scales with the density of the fluid so this contribution is small compared to the gas where density is typically 1000 times smaller. However, Boussinesq neglects this term to get:

$$\frac{D\theta}{Dt} = \alpha \nabla^2 \theta$$

If we non-dimensionalize with ΔT, d, d^2/α (time), then:

$$\mathrm{N-S}: \begin{cases} \dfrac{Du_i}{Dt} = -\dfrac{\partial p}{\partial x_i} + PrRa\Theta\delta_{i2} + Pr\nabla^2 u_i \\ \\ \dfrac{\partial u_j}{\partial x_j} = 0 \\ \\ u_i(x, y = \begin{matrix} 0 \\ 1 \end{matrix}, t) = 0 \ ; \ \text{rigid-rigid} \end{cases}$$

$$\mathrm{Temp}: \begin{cases} \dfrac{D\Theta}{Dt} = \nabla^2 \Theta \\ \\ \Theta(x, y = 0, t) = 0 \\ \\ \Theta(x, y = 1, t) = -1 \end{cases}$$

where

$$\Theta = \frac{T - T_0}{\Delta T}, \ Ra = \frac{g\beta\Delta T d^3}{\nu\alpha}, \quad Pr = \frac{\nu}{\alpha}$$

Ra = Rayleigh number: ratio of buoyancy to viscous forces. Note that

$$u_i = 0, \Theta = -y$$

corresponding to pure conduction. In the limit of zero viscosity this solution is unstable (infinite Rayleigh number). For diffusion present there is a certain amount that is required for stability; below that there is again an instability. This corresponds to a critical Rayleigh number Ra_c above which the conduction solution is no longer physically relevant. To analyze this we introduce infinitesimal disturbances to the *base state*, i.e.,

$$\begin{aligned} u_i &= \epsilon u_i' + \ldots \\ \Theta &= -y + \epsilon\theta' + \ldots \end{aligned}$$

Keeping only linear terms of order $\mathcal{O}(\epsilon)$ we obtain:

$$(1)\begin{cases} \dfrac{\partial \vec{v}'}{\partial t} = -\nabla p' + PrRa\theta'\hat{y} + Pr\nabla^2\vec{v}' \\ \nabla \cdot \vec{v}' = 0 \\ v'\left(x, y = \begin{matrix} 0 \\ 1 \end{matrix}, t\right) = 0 \end{cases}$$

$$(2)\begin{cases} \dfrac{\partial \theta'}{\partial t} - v' = \nabla^2\theta' \\ \theta'\left(x, y = \begin{matrix} 0 \\ 1 \end{matrix}, t\right) = 0 \end{cases}$$

Note that this linearized system predicts only whether natural convection will occur, but says nothing about the final state which corresponds to a finite amplitude response, obviously a nonlinear effect.

To simplify notation, we drop the primes in the following. In order to solve this system numerically we can either set up an eigenvalue problem or an initial value problem to discretize.

To obtain an eigenvalue problem we eliminate all unknowns but the vertical velocity component v. Taking the $\nabla\times$ of (1):

$$\frac{\partial \vec{\omega}}{\partial t} = RaPr(\nabla\theta \times \hat{y}) + Pr\nabla^2\vec{\omega}$$

and taking the $\nabla\times$ of this, we have:

$$\frac{\partial}{\partial t}\nabla^2\vec{v} = RaPr\left(\nabla^2\theta\hat{y} - \nabla\frac{\partial \theta}{\partial y}\right) + Pr\nabla^4\vec{v}$$

and the vertical component v:

$$\frac{\partial}{\partial t}\nabla^2 v = RaPr\frac{\partial^2\theta}{\partial x^2} + Pr\nabla^4 v$$

Also:

$$\frac{\partial \theta}{\partial t} - v = \nabla^2\theta$$

Introduce normal modes, i.e.,

$$\begin{aligned} v &= \tilde{v}(y)e^{ikx}e^{\sigma t} \\ \theta &= \tilde{\theta}(y)e^{ikx}e^{\sigma t} \end{aligned}$$

therefore:

$$\begin{aligned}\sigma(D^2-k^2)\tilde{v} &= -RaPrk^2\tilde{\theta}+Pr(D^2-k^2)^2\tilde{v}\\ \sigma\tilde{\theta}-\tilde{v} &= (D^2-k^2)\tilde{\theta}\Rightarrow(D^2-k^2-\sigma)\tilde{\theta}=-\tilde{v}\end{aligned}$$

where

$$D\equiv\frac{d}{dy}$$

or

$$\begin{cases}(D^2-k^2)[D^2-k^2-\sigma/Pr]\tilde{v}=k^2Ra\,\tilde{\theta}\\ (D^2-k^2-\sigma)\tilde{\theta}=-\tilde{v}\end{cases}$$

or

$$\begin{cases}(D^2-k^2)(D^2-k^2-\sigma/Pr)\tilde{v}=k^2Ra\tilde{\theta}\\ (D^2-k^2)(D^2-k^2-\sigma)(D^2-k^2-\sigma/Pr)\tilde{v}=-k^2Ra\,\tilde{v}\end{cases}$$

The equation for $\tilde{v}$ is of 6^{th}-order and thus we need 3 b.c. at each end. From $u=0$ at $y=$ const and continuity $\Rightarrow\frac{\partial v}{\partial y}=0$ so b.c.

$$\tilde{v}=\frac{\partial\tilde{v}}{\partial y}=\tilde{\theta}=0$$

on each rigid boundary or

$$\tilde{v}=D\tilde{v}=\tilde{\theta}=0$$

Using the first equation at the boundary we obtain:

$$D^4\tilde{v}-(2k^2+\sigma/Pr)D^2\tilde{v}=0$$

Note in general $\sigma=\sigma(k,Ra,Pr)$. However, the neutral stability is defined for $\sigma=0$ ($\sigma>0$, instability), and the above system is for the critical state:

$$(D^2-k^2)^3\tilde{v}=-k^2Ra_c\tilde{v}$$

$$\tilde{v}=D\tilde{v}=(D^2-k^2)^2\tilde{v}=0$$

thus, the critical state does not depend on Pr!

The solution consists of sines and exponentials but the eigenvalue relation is transcendental and thus it is difficult to present a simple explicit solution for $\sigma(k, Ra)$. A general (even) solution is of the form:

$$\begin{aligned}\tilde{v} = A_0 \cos\left[q_0\left(y-\frac{1}{2}\right)\right] &+ A\cosh\left[q\left(y-\frac{1}{2}\right)\right] \\ &+ A^*\cosh\left[q^*\left(y-\frac{1}{2}\right)\right]\end{aligned}$$

where

$$A_0 \,\epsilon\, \Re \; A, A^* \epsilon\, \mathcal{C}$$

$$\begin{aligned} q_0 &= k(\tau-1)^{1/2}, q = q_+ + iq_- \\ q_\pm &= k\left[\frac{1}{2}(1+\tau+\tau^2)^{1/2} \pm \left(1+\frac{1}{2}\tau\right)\right]^{1/2} \\ \tau &\equiv \left(\frac{Ra}{k^4}\right)^{1/3}\end{aligned}$$

To satisfy the b.c. we plug-in to get the eigenvalue relation:

$$Im\left\{\left(\sqrt{3}+i\right) q \tanh\frac{1}{2}q\right\} + q_0 \tan\frac{1}{2}q_0 = 0$$

Solution gives: $Ra_c \cong 1708$ for $k_c = 3.117$ and thus the (horizontal) wavelength of the disturbance at the onset of instability is

$$\frac{2\pi d}{k_c} = 2.016\ d$$

Numerical Formulation

Alternatively, we can solve an IVP by introducing the following expressions into the governing equations:

$$\left\{\begin{array}{c} u \\ v \\ p \\ \theta \end{array}\right\} = Re\left\{e^{ikx}\left[\begin{array}{c}\hat{u}(y,t) \\ \hat{v}(y,t) \\ \hat{p}(y,t) \\ \hat{\theta}(y,t)\end{array}\right]\right\}$$

and integrate in time for $\hat{u}, \hat{v}, \hat{p}, \hat{\theta}$.

We can choose any of the schemes we have studied so far. To decouple velocity/temperature we can use an explicit 2nd-order Adams-Bashforth. We then treat the remaining terms implicitly using a staggered grid where $\hat{u}, \hat{v}, \hat{\theta}$ are on the integer-point grid and $\hat{p}$ on the half-point grid. Crank-Nicolson will give an overall $\mathcal{O}(\Delta t^2)$ scheme, and Green's functions will give efficiency. The code can be written using complex arithmetic for convenience.

In the following, we give the discrete equations:

I. Convective Terms:

Use Adams Bashforth, 2nd-order:

$$
\begin{aligned}
&x\text{: } \hat{\hat{u}}_i^{n+1} = \hat{\hat{u}}_i^{n} \\
&y\text{: } \frac{\hat{\hat{v}}_i^{n+1} - \hat{v}_i^n}{\Delta t} = Pr \cdot Ra \left\{ \frac{3}{2}\hat{\theta}_i^n - \frac{1}{2}\hat{\theta}_i^{n-1} \right\} \\
&\text{Temp : } \frac{\hat{\hat{\theta}}_i^{n+1} - \hat{\theta}_i^n}{\delta t} = \frac{3}{2}\hat{v}_i^n - \frac{1}{2}\hat{v}_i^{n-1} \\
&\text{for } i = 0, \ldots, N
\end{aligned}
$$

Divergence after this step:

$$
\hat{\hat{d}}_{i+1/2}^{n+1} = ik\frac{\hat{\hat{u}}_{i+1}^{n+1} + \hat{\hat{u}}_i^{n+1}}{2} + \frac{\hat{\hat{v}}_{i+1}^{n+1} - \hat{\hat{v}}_i^{n+1}}{\Delta y}
$$

for $i = 0, \ldots, N'$.

For this splitting scheme used no b.c. are imposed on $\hat{\hat{u}}, \hat{\hat{v}}, \hat{\hat{\theta}}$ and continuity is not necessarily satisfied.

II. Stokes solver:

The momentum equations, energy equation and continuity discritized according to the suggested staggered mesh are:

$$
\begin{aligned}
x\text{: } \frac{1}{Pr}\frac{\hat{u}_i^{n+1} - \hat{\hat{u}}_i^{n+1}}{\Delta t} &= -ik\frac{p_{i+1/2}^{n+1/2} + p_{i-1/2}^{n+1/2}}{2} \cdot \frac{1}{Pr} \\
&+ \left(\frac{\hat{u}_{i-1}^{n+1/2}}{\Delta y^2} - \left(\frac{2}{\Delta y^2} + k^2 \right) \hat{u}_i^{n+1/2} + \frac{\hat{u}_{i+1}^{n+1/2}}{\Delta y^2} \right)
\end{aligned}
$$

$i = 1, \ldots, N'$ and $\hat{u}_0^{n+1} = \hat{u}_N^{n+1} = 0$

$$y\colon \frac{1}{Pr} \frac{\hat{v}_i^{n+1} - \hat{\hat{v}}_i^{n+1}}{\Delta t} = -\left(\frac{p_{i+1/2}^{n+1/2} - p_{i-1/2}^{n+1/2}}{\Delta y} \right) + \left[\frac{\hat{v}_{i-1}^{n+1/2}}{\Delta y^2} - \left(\frac{2}{\Delta y^2} + k^2 \right) \hat{v}_i^{n+1/2} + \frac{\hat{v}_{i+1}^{n+1/2}}{\Delta y^2} \right]$$

$i = 1, \ldots, N'$ and $\hat{v}_0^{n+1} = \hat{v}_N^{n+1} = 0$

Continuity:

$$\hat{d}_{i+1/2}^{n+1} \equiv ik \frac{\hat{u}_{i+1}^{n+1} + \hat{u}_i^{n+1}}{2} + \frac{\hat{v}_{i+1}^{n+1} - \hat{v}_i^{n+1}}{\Delta y} = 0$$

$$i = 0, \ldots N'$$

Temperature:

$$\frac{\hat{\theta}_i^{n+1} - \hat{\hat{\theta}}_i^{n+1}}{\Delta t} = \frac{\hat{\theta}_{i-1}^{n+1/2}}{\Delta y^2} - \left(\frac{2}{\Delta y^2} + k^2 \right) \hat{\theta}_i^{n+1/2} + \frac{\hat{\theta}_{i+1}^{n+1/2}}{\Delta y^2}$$

where

$$\hat{u}_i^{n+1/2} = \frac{\hat{u}_i^{n+1} + \hat{u}_i^n}{2};\; \hat{v}_i^{n+1/2} = \frac{\hat{v}_i^{n+1} + \hat{v}_i^n}{2}$$

$$\hat{\theta}_i^{n+1/2} = \frac{\hat{\theta}_i^{n+2} + \hat{\theta}_i^n}{2}$$

If we take $\frac{\partial}{\partial x}(M_x) + \frac{\partial}{\partial y}(M_y)$ we get:

$$\begin{aligned}
\frac{\hat{d}_{i+1/2}^{n+1} - \hat{\hat{d}}_{i+1/2}^{n+1}}{Pr\Delta t} &= \left\{ -\frac{1}{\Delta y^2} \left(p_{i+3/2}^{n+1/2} - 2p_{i+1/2}^{n+1/2} + p_{i-1/2}^{n+1/2} \right) \right. \\
&+ \left. \frac{k^2}{4} \left(p_{i+3/2}^{n+1/2} + 2p_{i+1/2}^{n+1/2} + p_{i-1/2}^{n+1/2} \right) \right\} \\
&+ \frac{1}{2} \left(\frac{\hat{d}_{i-1/2}^{n+1}}{\Delta y^2} - \left(\frac{2}{\Delta y^2} + k^2 \right) \hat{d}_{i+1/2}^{n+1} + \frac{\hat{d}_{i+3/2}^{n+1}}{\Delta y^2} \right) \\
&+ \frac{1}{2} \left(\frac{\hat{d}_{i-1/2}^{n}}{\Delta y^2} - \left(\frac{2}{\Delta y^2} + k^2 \right) \hat{d}_{i+1/2}^{n} + \frac{\hat{d}_{i+3/2}^{n}}{\Delta y^2} \right)
\end{aligned}$$

Assume that at the time step $(n+1)$ continuity is already satisfied for previous time step, therefore $d^n_{i+1/2} = 0$. Then if we satisfy the discrete momentum equation and the Poisson equation:

$$\left\{ \overbrace{\frac{1}{\Delta y^2}\left(p^{n+1/2}_{i+3/2} - 2p^{n+1/2}_{i+1/2} + p^{n+1/2}_{i-1/2}\right) - \frac{k^2}{4}\left(p^{n+1/2}_{i+3/2} + 2p^{n+1/2}_{i+1/2} + p^{n+1/2}_{i-1/2}\right)}^{Lp^{n+1/2}_{i+1/2}} \right\}$$

$$= \frac{\hat{\hat{d}}^{n+1}_{i+1/2}}{\Delta t Pr}$$

the remaining parts give the equation:

$$\frac{\hat{d}^{n+1}_{i+1/2}}{Pr\Delta t} = \frac{1}{2}\left(\frac{\hat{d}^{n+1}_{i-1/2}}{\Delta y^2} - \left(\frac{2}{\Delta y^2} + k^2\right)\hat{d}^{n+1}_{i+1/2} + \frac{\hat{d}^{n+1}_{i+3/2}}{\Delta y^2}\right), i = 1, \ldots, N' - 1$$

$$\hat{d}^{n+1}_{1/2} = \hat{d}^{n+1}_{N'+1/2} = 0$$

with the trivial solution $\hat{d}^{n+1}_{i+1/2} = 0$. Now, the full scheme is as follows:

I. Solve for $\hat{\hat{u}}, \hat{\hat{v}}, \hat{\hat{\theta}}$ using A/B (E-F as start up)

Form the divergence $\hat{\hat{d}}^{n+1}_{i+1/2}$

II. Solve

$$\mathcal{L}p^{n+1/2}_{i+1/2} = \frac{\hat{\hat{d}}^{n+1}_{i+1/2}}{Pr\Delta t} \quad i = 1, \ldots, N' - 1$$

b.c.

$$\hat{d}^{n+1}_{1/2} = 0 \quad \hat{d}^{n+1}_{N'+1/2} = 0$$

III. For all interior points $i = 1, \ldots, N'$ solve

$$\frac{\hat{\hat{\hat{u}}}^{n+1}_i - \hat{\hat{u}}_i}{Pr\Delta t} = -ik\frac{p^{n+1/2}_{i+1/2} + p^{n+1/2}_{i-1/2}}{2} \equiv -D_x p^{n+1/2}_{i+1/2}$$

$$\frac{\hat{\hat{\hat{v}}}^{n+1}_i - \hat{\hat{v}}^n_i}{Pr\Delta t} = -\left(\frac{p^{n+1/2}_{i+1/2} - p^{n+1/2}_{i-1/2}}{\Delta y}\right) = -D_y p^{n+1/2}_{i+1/2}$$

IV. For all interior points $i = 1, \ldots, N'$ solve

$$\left(\frac{\hat{u}_{i-1}^{n+1/2}}{\Delta y^2} - \left(\frac{2}{\Delta y^2} + k^2 + \frac{2}{Pr\Delta t}\right)\hat{u}_i^{n+1/2} + \frac{\hat{u}_{i+1}^{n+1/2}}{\Delta y^2}\right) = \frac{-\hat{u}_i^n - \hat{\hat{u}}_i^{n+1}}{Pr\Delta t}$$

$$\hat{u}_0 = \hat{u}_N = 0$$

$$\left(\frac{\hat{v}_{i-1}^{n+1/2}}{\Delta y^2} - \left(\frac{2}{\Delta y^2} + k^2 + \frac{2}{Pr\Delta t}\right)\hat{v}_i^{n+1/2} + \frac{\hat{v}_{i+1}^{n+1/2}}{\Delta y^2}\right) = \frac{-\hat{v}_i^n - \hat{\hat{v}}_i^{n+1}}{Pr\Delta t}$$

$$\hat{v}_0 = \hat{v}_N = 0$$

The scheme, now, proceeds using the Green's function implementation.

I. Pre-Procession: After we solve the Laplace equation for pressure, the RHS of the Helmoltz solvers for U_i^k, V_i^k can be formed. The coefficients for the Helmoltz operator are:

$$\frac{1}{\Delta y^2}, -\left(\frac{2}{\Delta y^2} + k^2 + \frac{2}{Pr \cdot \Delta t}\right), \frac{1}{\Delta y^2}$$

II. Time-stepping-Intermediate Fields:

- Solve discrete Poisson equation:

$$\mathcal{L}P_{i+1/2}^I = \frac{\hat{\hat{d}}_{i+1/2}^{n+1}}{Pr \cdot \Delta t}$$

- Update velocities:

$$\frac{\hat{U}_i^I - \hat{\hat{u}}_i^{n+1}}{Pr\Delta t} = -D_x P_{i+1/2}^I$$

$$\frac{\hat{V}_i^I - \hat{\hat{v}}_i^{n+1}}{Pr \cdot \Delta t} = -D_y P_{i+1/2}^I$$

- Viscous step:

$$\left(\frac{2}{\Delta y^2} - \left(\frac{2}{\Delta y^2} + k^2 + \frac{2}{Pr\Delta t}\right) + \frac{2}{\Delta y^2}\right) U_i^I = -\frac{\hat{u}_i^n + \hat{U}^I}{Pr\Delta t}$$

with b.c.

$$\begin{aligned} U_0^I &= U_N^I = 0 \\ V_0^I &= V_N^I = 0 \end{aligned}$$

Apply superposition:

$$\hat{u}_i^{n+1/2} = U_i^I + \alpha_1 U_i^1 + \alpha_2 U_i^2 \text{ etc.}$$

and then find:

$$\hat{u}_i^{n+1} = 2\hat{u}_i^{n+1/2} - \hat{u}_i^n, \text{ etc.}$$

Similarly for temperature:

$$\frac{\theta_i^{n+1} - \hat{\theta}_i^{n+1}}{\Delta t} = \nabla^2 \frac{\theta_i^{n+1} + \theta_i^n}{2} \equiv \nabla^2 \theta^{n+1/2}$$

or

$$\frac{2\left(\frac{\theta_i^{n+1}+\theta_i^n}{2}\right) - 2\hat{\theta}_i^{n+1}}{\Delta t} = \nabla^2 \theta_i^{n+1/2}$$

or

$$\left(\nabla^2 - \frac{2}{\Delta t}\right)\theta_i^{n+1/2} = \frac{-2}{\Delta t}\hat{\theta}_i^{n+1}$$

and then find

$$\theta_i^{n+1} = 2\theta_i^{n+1/2} - \theta^n, \quad i = 1, \ldots, N'$$

Question (a): The stability curve is shown in figure 1. The resolution used was 21 grid points, $\Delta t = 0.01$ and final real time $= 2.0$. The min $Ra_{\text{cr}} \cong 1721.5$ and $k_{\text{cr}} = 3.10/3.20$ are compared with the exact numbers given by analytical work $k = 3.17$, $Ra = 1708$ (Reference Physical Fluid Dynamics, D.J. Tritton, pg. 216). The locus of $\sigma = 0$ separating the stable region with $\sigma < 0$ from the unstable region with $\sigma > 0$ is independent of the Prandtl number. This is verified numerically but also can be anticipated since the inertia term $\frac{Du}{Dt}$ is zero at this curve since both the temporal acceleration $\frac{\partial u}{\partial t}$ and the convective term is zero because there is no mean velocity field and second order terms in the perturbation velocity field are neglected.

(b) In figure 2 the asymptotic value of σ (the eigenvalue) as a function of the resolution is given. The value of σ predicted to be $\sigma = 4.3365$. (Final real time was 2.5, $\Delta t = 0.01$)

Rayleigh-Bernard Problem

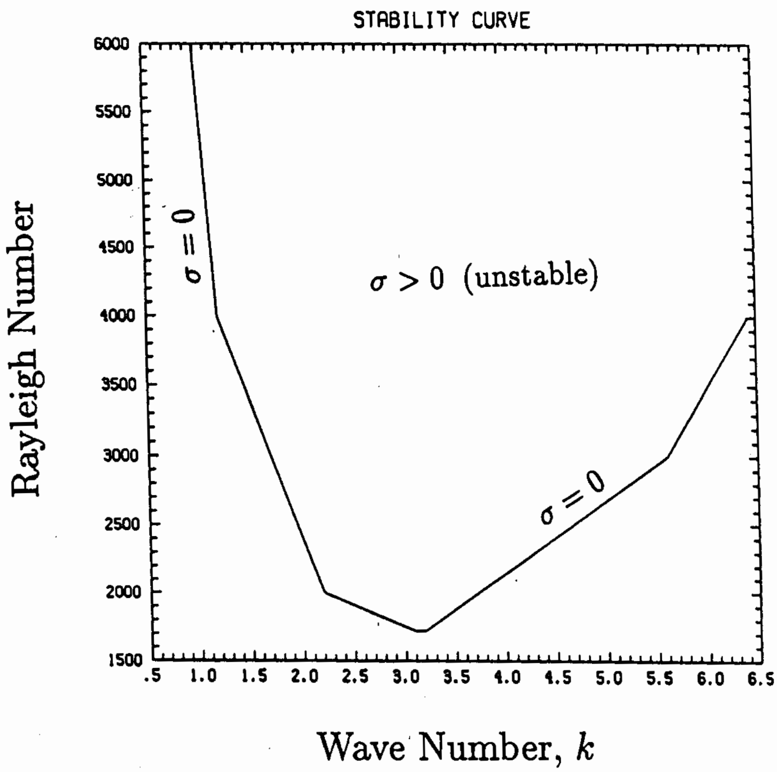


Figure 1

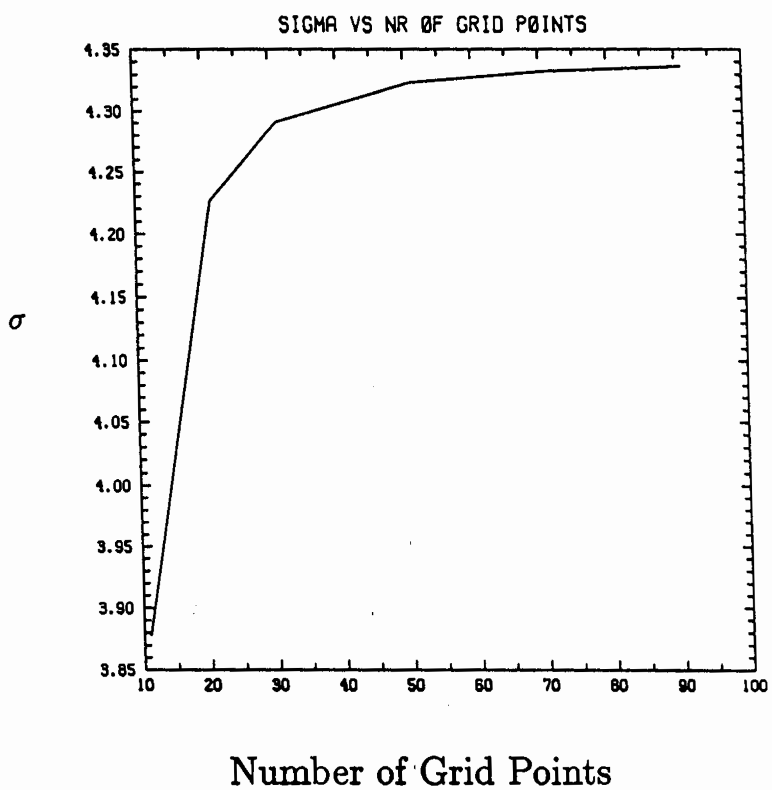


Figure 2