AM 258

Computational Fluid Dynamics

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Homework #3

Consider the first-order linear ODE:

$$\frac{dy}{dt} = -ky \qquad \text{with } y\,(t=0) = y_0 \text{ and } t \in [0, T]\,, \tag{1}$$

where k is a stochastic process of second order such that:

$$k = \overline{k} + \nu\,(t, \omega)\,. \tag{2}$$

Here $\overline{k}$ is the mean value of k and $\nu\,(t, \omega)$ represents a random variable depending on time and random space. We assume that the probability distribution function (PDF) remains the same for all k's and takes the form of a Gaussian distribution with constant variance. The PDF of k at a fixed time t is:

$$f\,(k\,(t)) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\left(k(t)-\overline{k}\right)^2}{\sigma^2}} \tag{3}$$

where $\overline{k}$ is the mean value of k and σ^2 is the variance of k.

Random values of k, generated in time every Δt_s can be mutually independent, partially correlated or fully correlated. To this end, we consider the following cases:

Case 1**:** The PDF of the solution for the *mutually independent* case takes the form:

$$f\,(y\,(t)) = \frac{1}{\sigma y\sqrt{2\pi t\Delta t_s}} e^{-\frac{1}{2}\frac{\left(\ln\frac{y}{y_0}+\overline{k}t\right)^2}{\sigma^2 t\Delta t_s}} \tag{4}$$

Case 2**:** The PDF of the solution for the partially correlated case takes the form:

$$f\,(y\,(t)) = \frac{1}{\sigma\Delta t_s S y\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\left(\ln\frac{y}{y_0}+\overline{k}t\right)^2}{(\sigma\Delta t_s S)^2}}, \quad \Delta t_s \ll T \tag{5}$$

and

$$S = \left(N\frac{(1+C)}{(1-C)} - 2C\frac{(1-C^N)}{(1-C)^2}\right)^{\frac{1}{2}} \tag{6}$$

where $C = e^{-\frac{\Delta t_s}{A}}$, $N = \frac{T}{\Delta t_s}$ and A is the correlation length of the random process.

Case 3: In the fully correlated case, the random process k becomes a random variable and the PDF of the solution takes the form:

$$f\left(y\left(t\right)\right) = \frac{1}{\sigma t y\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\left(\ln\frac{y}{y_0}+\overline{k}t\right)^2}{(\sigma t)^2}} \tag{7}$$

An easy way to write the first moment in a general form for all cases is to express it in a logarithmic form:

$$\ln\left(\frac{E\left(y\right)}{y_0}\right) = -\overline{k}t + \frac{1}{2}\sigma^2\Gamma \tag{8}$$

where $\Gamma = t\Delta t_s$ for Case 1, $\Gamma = t^2$ for Case 3 and $\Gamma = \Delta t_s\left(t\frac{(1+C)}{(1-C)} - 2C\Delta t_s\frac{(1-C^N)}{(1-C)^2}\right)$ for Case 2.

Second moments for all cases can be obtained from the expressions of the mean.

$$E\left(y^2\right) = \left(e^{\sigma^2\Gamma} - 1\right)E^2\left(y\right) \tag{9}$$

where $\Gamma = t\Delta t_s$ for Case 1, $\Gamma = t^2$ for Case 3 and $\Gamma = t\Delta t_s\left(t\frac{(1+C)}{(1-C)} - 2C\Delta t_s\frac{(1-C^N)}{(1-C)^2}\right)$ for Case 2.

(a) Use a multi-step or a multi-stage ODE solver of various orders to obtain solutions and corresponding errors for the mean and variance response corresponding to $\bar{k} = 0; \sigma = 1$, and final time of integration $T = 1$. To do this you need to follow a **Monte Carlo** approach where $\nu(t,\omega)$ is obtained from a Gaussian PDF corresponding to $\sigma = 1$. The correlation length for Case 2 is assumed to be $A = 0.1$.

(b) Does the accuracy of the numerical solutions you obtained increase with the formal order of the time-stepping scheme you used as expected? Explain.

Note: Please use the following formula to update the fluctuating part of the coefficient:

$$\nu^{n+1} = C\nu^n + \sigma\sqrt{1-C^2}\xi,$$

where ξ is a Gaussian random variable.