REPORT OF ENTROPY-BASED NONLINEAR VISCOSITY FOR CONSERVATION LAWS

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Jean-Luc Guermond et.al proposed an entropy-based nonlinear viscosity ([2, 3]) to solve hyperbolic equations.

1. Algorithm

We consider the hyperbolic equation

$$\partial_t u + \nabla \cdot \boldsymbol{f}(u) = 0, \qquad u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, t > 0 \tag{1}$$

subject to appropriate boundary conditions. It is well know that Cauchy or the initial boundary value problem has a unique entropy solution satisfying

$$\partial_t E(u) + \nabla \cdot \boldsymbol{F}(u) \leq 0, \tag{2}$$

where entropy $E(u)$ is a convex function and $\boldsymbol{F}(u) = \int E'(u)\boldsymbol{f}'(u)\mathrm{d}u$ is the entropy flux. The idea of the entropy-based nonlinear viscosity is to construct viscosity through the entropy residual:

$$D(x,t) = \partial_t E(u(x,t)) + \nabla \cdot \boldsymbol{F}(u(x,t)), \boldsymbol{x} \in \Omega, t > 0. \tag{3}$$

Let $u_h(\cdot, t)$ be the numerical approximation of the exact solution u at time t (and similarly subscript $_h$ denote the approximation of the variables). The entropy viscosity method comprises of the following steps ([3]):

(1) Given an entropy pair $(E, \boldsymbol{F})$, define the entropy residual:
$$D(x,t) = \partial_t E(u(x,t)) + \nabla \cdot \boldsymbol{F}(u(x,t)), \boldsymbol{x} \in \Omega, t > 0.$$
(2) Use this residual to define a viscosity, say ν_E
$$\nu_E(\boldsymbol{x}, t) = c_E h^2(\boldsymbol{x}) R(D_h(\boldsymbol{,} t))/\|E(u_h) - \bar{E}(u_h)\|_{\infty,\Omega},$$
where $h(\boldsymbol{x})$ is the local mesh size at $\boldsymbol{x} \in \Omega$, $\bar{E}$ is the space-averaged value of the entropy, c_E is a tunable constant and R is a positive function to be decided ($R(D_h) = |D_h|$ in this report and also in [2, 3]).
(3) Introduce an upper bound to the entropy viscosity:
$$\nu_{\max}(\boldsymbol{x}, t) = c_{\max} h_{\max} \max_{\boldsymbol{y} \in V_x} |\boldsymbol{f}'(u(\boldsymbol{y}, t))|.$$
Here V_x is a yet to be defined neighborhood of $\boldsymbol{x}$, $\boldsymbol{f}'(u(\boldsymbol{y}, t))$ is the local wave speed.
(4) Define the entropy viscosity:
$$\nu_h = S(\min(\nu_{\max}, \nu_E)),$$
where S is a yet to be defined smoothing operator that depends on the space approximation (the simplest case is $S = I$).
(5) Augment the discrete form of the conservation law (1) with the dissipation term $-\nabla \cdot (\nu_h \nabla u_h)$ and make the viscosity explicit.

To conclude the equation we in fact solve is the advection-diffusion equation with artificial viscosity:

$$\partial_t u + \nabla \cdot \boldsymbol{f}(u) = \partial_x(\nu(u)\partial_x u). \tag{4}$$

At time step t_{k+1}, the viscosity ν is made explicit and evaluated at time t_k.

This simple idea is *mesh and approximation independent* and can be applied to any equation or physical system supplemented with an entropy equation/inequality.

For time integral, the semi-discretized equation

$$\frac{\mathrm{d}}{\mathrm{d}t}U = L(U)$$

is solved with 3rd-order TVD Runge-Kutta (also called SSP Runge-Kutta [1]). To approximate $\partial_t E(u)$, second order finite difference is used:

$$\partial_t E^n \approx \frac{3E(u^n) - 4E(u^{n-1}) + E(u^{n-2})}{2\Delta t}, \qquad n \geq 2.$$

For $n = 1$ first order finite difference is used and for $n = 0$ let $\partial_t E = 0$.

2. Computational results

In this section we present two 1-D computational results:

- Burges equation with shock wave fully developed:

$$\partial_t u + \partial_x(u^2/2) = 0, \quad x \in [0, L]$$
$$u(x, 0) = \sin(2\pi x/L).$$

  The final time is $T = L/4$, which is 1/4 period.
- Long time evolution of transport equation.

$$\partial_t u + \partial_x u = 0, \quad x \in [0, 1]$$

  with initial condition

$$u(x,0) = \begin{cases} \exp(-300(2x-0.3)^2) & |2x-0.3| \leq 0.25, \\ 1 & |2x-0.9| \leq 0.2, \\ \left(1 - (\frac{2x-1.6}{0.2})^2\right)^{1/2} & |2x-1.6| \leq 0.2, \\ 0 & \text{otherwise}. \end{cases}$$

  The final time is $T = 100$ which is 100 period.

Figure 1 shows the initial condition and the final results for these two problems.

2.1. Fourier collocation method.

For 1-D Burges equation, we set $E(u) = u^2/2, R(D) = |D|, S = I$. The result is shown in Fig. 2 Table 1 shows the convergence rates in L_1 and L_2 norm. We can see that for discontinuous problem the convergence rate in L_1 norm is 1 and 0.5 in L_2 norm.

TABLE 1. L_1 and L_2 error and the convergence rate of the solution of Burges equation

h	L_1	rate	L_2	rate
$2\pi/100$	1.551e-1	-	2.715e-1	-
$2\pi/200$	8.095e-2	0.94	1.967e-1	0.46
$2\pi/400$	4.305e-2	0.91	1.408e-1	0.48
$2\pi/800$	2.168e-2	0.99	9.928e-2	0.50

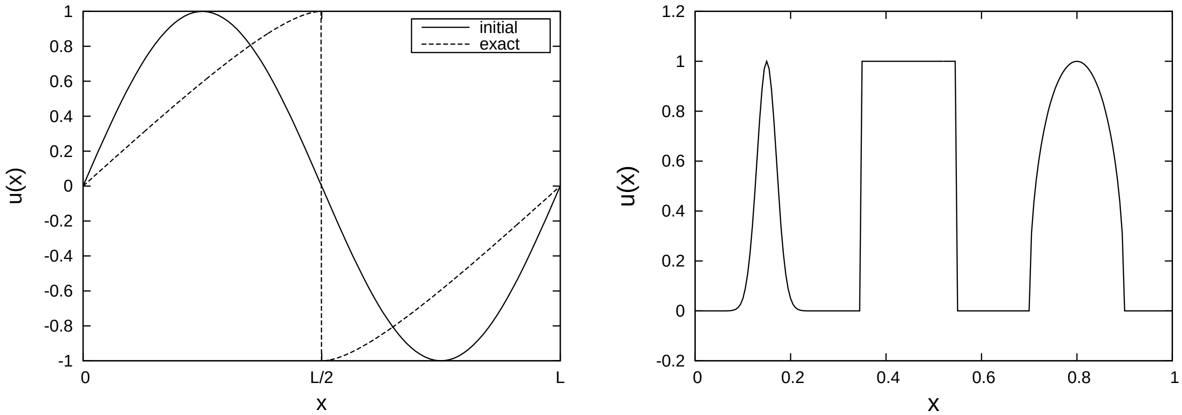

FIGURE 1. Initial condition and exact solution for Burges equation (left) on $[0, L]$ at $T = L/4$ and transport equation (right) on $[0, 1]$ at $T = 100$. For the transport equation the exact solution is the same as the initial condition so only one curve is shown.

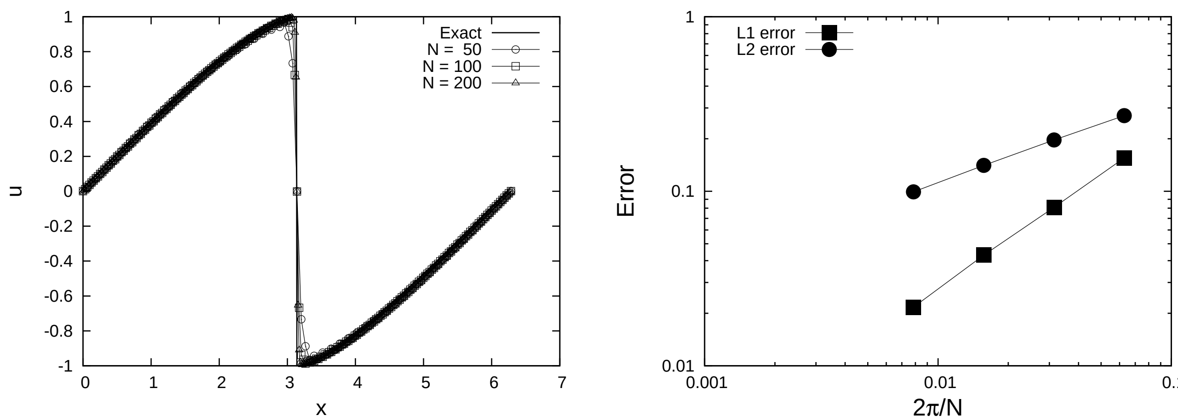

FIGURE 2. Left: Solution of Burges equation on $[0, 2\pi]$. $t = \pi/2$, $\alpha_{\max} = 2/\pi$, $\alpha = 0.1$. Right: L_1 and L_2 error of the solution of Burges equation on $[0, 2\pi]$ with different h.

2.2. **Spectral element method.** The spectral element shape functions are the Lagrange polynomials based on the $k+1$ Gauss-Lobatto-Legendre points in 1D where k is the order of the polynomials. The quadrature points are based on the Gauss-Lobatto-Legendre points so that the interpolation points and quadrature points coincide. We compute the solution at $t = 100$, with $k = 2, 4, 8$. The mesh is composed of $200/k$ cells so that the total number of degrees of freedom is 200. The parameters are set as $c_{\max} = 0.1/k, c_E = 1.0, \Delta t = 0.1 h_{\min}$. We can see from left plot of Figure 3 that if we do not include viscosity there will be severe oscillations even with polynomial of order 8. In the right plot of Figure 3 results by viscosity is shown. We can observe that there is no oscillations in the result by using entropy-based viscosity.

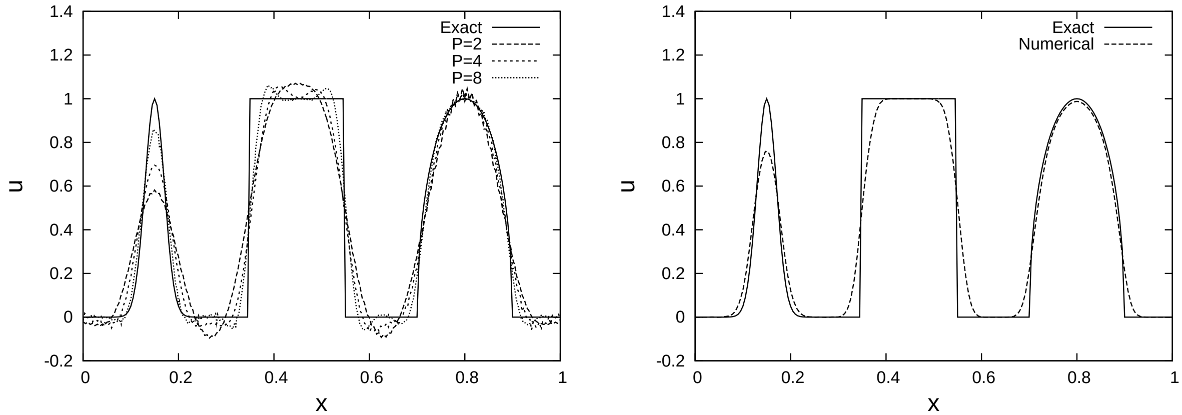

FIGURE 3. Solution of transport equation without viscosity (left) and with entropy-based viscosity (right). In the right plot $k = 8$.

REFERENCES

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