INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng. 2002; **53**:1763–1764 (DOI: 10.1002/nme.363)

BOOK REVIEW

DISCONTINUOUS GALERKIN METHODS: THEORY, COMPUTATION AND APPLICATION (LECTURE NOTES IN COMPUTATIONAL SCIENCE AND ENGINEERING), by B. Cockburn, G. E. Karniadakis and C.-W. Shu (eds), Springer, Berlin, 2000. ISBN 3-540-66787-3 GB£51.50.

In the last few years the name *Discontinuous Galerkin*, as a general methodology, has become very popular and contributions have been made by many authors. However, the very origins of the name, and indeed of the procedure as currently practiced, are much older. They were first introduced as a treatment of pure scalar-convection problems by Reed and Hill (1973) and Lesaint and Raviart (1974). Later others extended the original procedures to dealing with first order equations in time and achieved some success there. The recent developments, however, are much more important and of much larger scope. The objective outlined by the editors is to present a coherent picture of research currently being made in this field and in this they achieve their objective.

The book starts with a long (40 plus page) paper by the editors in which the current position of the discontinuous-Galerkin method is summarized and which also discusses the content of the subsequent papers. The remaining part of the book is divided into two parts. The first part contains 16 invited papers by well-known researchers in this field many of whose names will be recognizable to finite element practitioners from their contributions in other areas. The last section contains 32 contributed papers which also present a good picture of current research.

The recent interest is, in the reviewers belief, caused by the extension of the discontinuous-Galerkin method to diffusion equations as well as to the pure convection ones. Only through this large extension is it possible to reach the full range of applications necessary for adequate solution of fluid mechanics problems and other general finite-element problems.

It appears to the reviewers that the presentation of the introductory article, and indeed of the subsequent papers, lacks a simple introduction to the nature of what is currently described as the discontinuous-Galerkin method and, perhaps, it will serve some useful purpose to give this at the beginning of the review. It appears to us that the methodology is one in which separate domains, or elements exist, in each of which h, p, or spectral discretization is used following the standard-Galerkin procedures. The *discontinuity* is one arising from the attempt to link the various domains and, though classically this has been done many times by the introduction of the Lagrangian variables, here a desire and need exists to eliminate the presence of the Lagrange multipliers. In the discontinuous-Galerkin method the Lagrange multipliers on the interface between the various subdomains or elements are eliminated by their identification with the quantities contributed appropriately from each domain.

Quite early in the finite-element literature it was found that such direct substitution on most occasions will fail unless some stabilization is added. Here important theoretical contributions have been made by Nitsche in 1971, where explicit continuity of the variable is made by a variational process. Similar work was practically applied by Kikuchi and Ando in 1972 to plate and shell problems.

Probably the most useful discussion of the discontinuous-Galerkin procedure has been done recently by Oden and Baumann and their work, some of which is presented in the present volume, outlines the possibilities very clearly. Others have presented somewhat similar procedures, and a paper by Arnold *et al.* summarizes current theory succinctly. The work by Oden and Baumann and even more importantly the work of Karniadakis *et al.* is most impressive. Both succeed in solving problems in fluid mechanics using very high-order element expansions for the linked subdomains. The reviewers are not aware of the actual order presented in each sub-domain but it is believed from the illustrations that quite high orders are pursued and their work is probably the first in which higher-order approximations have been introduced to the area of compressible fluid mechanics.

Why has it taken so long to use higher-order approximations in fluid mechanics? The reviewers believe that the difficulty lies in the fact that iterative methods have generally needed to be used and, indeed, have always been used for successful solutions with the simplest elements or finite volumes. These methods use explicit time integration and generally such explicit methods cannot be applied to a large number of variables or linked by a continuous approximation which employs arbitrary p-order interpolations. The discontinuous-Galerkin method permits the evaluation of a mass matrix for transient problems in each domain separately. And indeed this mass matrix can be either made directly diagonal by a suitable choice of shape functions, as is summarized in the volume by Karniadakis *et al.,* or more generally by a local numerical solution.

It is of interest to carry out some model experiments to see how well the contribution of discontinuous-Galerkin increases the overall accuracy if at all. Clearly, a very large penalty is paid in comparison with direct use of finite elements as the number of variables is dramatically increased. This last point is only mentioned briefly in one paper by Hughes *et al.* which includes a table showing the increase in the number of variables depending on the type of element being used. This matter which has to be taken into serious consideration and, of course, the finite-element procedure in general is enriched by the processes outlined in this book.

We would like to conclude this review by citing the words of the editors who state ‘the discontinuous-Galerkin process at last allows many of the outdated finite-difference and finite-volume procedures to be abandoned’. It is high time to do this in view of the increased accuracy invariably presented by finite elements and the rather prejudiced way in which finite elements are viewed by the finite-volume or finite-difference protagonists. The preference of many finite-difference and finite-volume practitioners is for the local conservativity which they imagine is abandoned in the finite-element process. This clearly is not true, as any finite-element user will confirm, as in finite elements better values of flux quantities at any point always will be obtained by a suitable recovery.

O. C. Zienkiewicz
Department of Civil Engineering
University of Wales, *Swansea*,
Swansea SA2 8PP, *U.K.*

R. L. Taylor
University of California
Berkeley, *CA*, *U.S.A.*

